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Foreword

The present publication, “Hedging versus Insurance: Long-Horizon Investing with Short-Term Constraints,” was produced as part of the BNP Paribas Investment Partners research chair at EDHEC-Risk Institute on “ALM and Institutional Investment Management”. Under the supervision of Professor Lionel Martellini, Scientific Director of EDHEC-Risk Institute, this chair examines the properties of dynamic asset allocation strategies in asset-liability management.

Indeed, this document provides comprehensive insights into all of EDHEC-Risk Institute’s research on dynamic allocation in asset-liability management.

The publication builds on these previous findings and illustrates that failing to separate long-term risk-aversion and short-term loss-aversion may lead to poor investment decisions. Relatively simple solutions exist that can be implemented as dynamic asset allocation strategies in order to control short-term risk levels while maintaining access to long-term sources of performance. These solutions are a substantial improvement over traditional strategies without dynamic risk-control, which inevitably lead to under-spending of investors’ risk budgets in normal market conditions, with a strong associated opportunity cost, and over-spending of investors’ risk budget in extreme market conditions.

I would like to thank the co-authors of the publication, Romain Deguest, Lionel Martellini and Vincent Milhau, for their groundbreaking research on dynamic allocation in an asset-liability management setting.

I would also like to extend our warmest thanks to BNP Paribas Investment Partners for their generous support and for their continuing commitment to this research chair.

We wish you a pleasant and informative read.

Noël Amenc
Professor of Finance
Director of EDHEC-Risk Institute
About the Authors

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Executive Summary
Executive Summary

Designing investment solutions that meet investors’ needs requires addressing the conflict inherent to the coexistence of long-term objectives and short-term constraints.

Meeting the challenges of modern investment practice involves the design of novel forms of investment solutions, as opposed to investment products, customised to meet investors’ long-term objectives, while respecting a number of constraints expressed in terms of dollar budgets, but also risk budgets which are often set over short horizons.

The conflict between the presence of long-term objectives and short-term constraints is in fact one of the most critical challenges faced by long-term investors. Focusing on equity investments, the dilemma can be summarised as follows. On the one hand, investing substantial fractions of their wealth in equity markets makes it difficult for investors to ensure the respect of short-term risk budgets, and leads to a dominant allocation to safe assets that show a better correlation with the investors’ liabilities. On the other hand, shying away from investing in equity involves a substantial opportunity cost, especially for long-term investors, because the equity risk premium is positive, hence attractive for all investors, but also mean-reverting, hence even more attractive for long-term investors.

One of the main findings in the academic literature on long-term allocation decisions with mean-reverting equity returns is the fact that a long-term allocation to equities serves as a hedge against unfavourable short-term equity returns in the presence of mean-reverting equity returns. As a result, the optimal allocation to stocks is higher compared to the myopic case, and investors with longer time-horizons hold more stocks compared to investors with shorter horizons. This prescription has very often been taken at face value by target date funds or life-cycle funds an investment solution advocating a deterministic decrease of equity allocations (also known as glidepath) when approaching retirement date.

One key problem, however, is that this prescription can lead to extremely difficult situations when risk is assessed from a shorter-term perspective, in particular in the context of a severe bear equity market such as the one experienced in 2008. In fact, it appears that most if not all investors, even those with the longest possible horizons (such as pension funds or sovereign wealth funds), inevitably face a number of short-term performance constraints, imposed by accounting and/or regulatory pressure, political pressure, peer pressure, etc. In a private wealth management context, there is also strong evidence that investors typically face (mostly self-imposed) short-term constraints, e.g., maximum drawdown constraints.

While it is widely perceived that a tension exists between a focus on hedging long-term risk and a focus on insurance with respect to short-term constraints, we cast new light on this debate by arguing that long-term objectives and short-term constraints need not be mutually exclusive. In fact, our analysis shows that both motives may naturally coexist within the context of a long-term investing strategy consistent with short-term performance constraints.
Executive Summary

In the absence of short-term risk constraints, the presence of a mean-reverting equity risk premium justifies that the allocation to equities should be anti-cyclical and increasing with the time-horizon.

Investors endowed with consumption/liability objectives need to invest in two distinct portfolios, in addition to cash: one performance-seeking portfolio (PSP) and one liability-hedging portfolio (LHP); this is the liability-driven investing (LDI) paradigm. The allocation to the “risky” performance-seeking portfolio (PSP) versus “safe” liability-hedging portfolio (LHP) is found to be increasing in the PSP Sharpe ratio.

As such the optimal strategy displays a state-dependent component, suggesting that the allocation to equity should be increased (respectively, decreased) when equity has become cheap (respectively, expensive), as measured through a proxy for the equity risk premium. In the context of a model with a stochastic mean-reverting equity risk premium, one can also show that the optimal allocation involves also a hedging demand against unexpected changes in the PSP Sharpe ratio, known as risk premium hedging portfolio (RPHP), which implies indeed a deterministic decrease of the allocation to equity as the investor gets closer to the time-horizon.

One key element that is missing in the analysis presented so far is the integration of short-term constraints into the design of the optimal allocation strategy.

The presence of short-term constraints justifies that the allocation to equities should also be an increasing function of the current value of the short-term risk budget, in addition to being an increasing function of the time-horizon and the current value of the equity risk premium.

These short-term constraints are not managed through hedging strategies, which focus on immunising the portfolio value with respect to changes in risk factors impacting asset and liability values, but instead through dedicated insurance strategies. The practical implication of the introduction of short-term constraints is that optimal investment in a performance-seeking satellite portfolio (PSP) is not only a function of risk aversion, but it also becomes a function of risk budgets (margin for error measured in terms of distance with respect to minimum acceptable wealth levels), and of the probability of the risk budget being spent before horizon. In a nutshell, a pre-commitment to risk management allows one to adjust risk exposure in an optimal state-dependent manner, and therefore to generate the highest exposure to upside potential of PSP while respecting risk constraints.

It is widely perceived that a tension exists between a focus on hedging long-term risk and a focus on insurance with respect to short-term constraints: typically, dynamic risk-controlled strategies, which imply a reduction to equity allocation when a drop of equity prices has led to a substantial diminution of the risk budget, have often been blamed for their pro-cyclical nature, and long-term investors are often reluctant to sell equity holdings in those states of the
world where equity markets have become particularly attractive in the presence of mean-reversion in the equity risk premium.

Our research actually suggests that long-term objectives and short-term constraints need to be mutually exclusive, and can be integrated in a comprehensive asset allocation framework. Depending on market conditions and parameter values, the pro-cyclical risk-controlled motivation may outweigh the revision of strategic asset allocation motivation, or vice-versa, with risk management always ultimately prevailing. In other words, the risk-control methodology can be made entirely consistent with internal or external processes aiming to generate active asset allocation views. In fact, casting the active view generation process within the formal framework of a dynamic risk-control strategy appears to be the only way to successfully implement active asset allocation decisions while ensuring the respect of risk limits.

In practice, a number of key improvements can be used in implementation. While the original approach was developed in a simple framework, it can be extended in a number of important directions, allowing for the introduction of more complex floors. A large variety of floors can in fact be introduced (simultaneously if necessary) so as to accommodate the needs of different kinds of investors. Among the possible floors, the following possibly stand out in terms of their relevance for various kinds of investors: capital guarantee floors allowing for the protection of a fraction of the initial capital; benchmark protection floors allowing for the protection of a fraction of the value of any given stochastic benchmark (with the liability portfolio being the most natural benchmark for investors facing liabilities); max drawdown floors allowing for the respect of limits on maximum consecutive losses; trailing performance floors allowing for the protection of a fraction of the prior value of the portfolio on a rolling basis; etc.

In addition to accounting for the presence of floors, the dynamic risk-controlled strategies can also accommodate the presence of various forms of caps or ceilings. These strategies recognise that the investor has no utility over a cap target level of wealth, which represents the investor’s goal (actually a cap), which can be a constant, deterministic or stochastic function of time. From a conceptual standpoint, it is not clear a priori why any investor should want to impose a strict limit on upside potential. The intuition is that by forgoing performance beyond a certain threshold, where they have relatively lower utility from higher wealth, investors benefit from a decrease in the cost of the downside protection (short position in a convex payoff in addition to the long position – collar flavour).

Putting it differently, without the performance cap, investors have a greater chance of failing an almost attained-goal when their wealth level is very high, and we show that the presence of upper (in addition to lower) bounds on performance, consistent with the kind of utility satiation often exhibited by long-term investors, is another, independent, reason why a fall in equity prices should not always lead to a decrease of equity allocation, even without mean-reverting equity risk premium.

The opportunity costs implied by the short-term constraints are significantly lower when these constraints are
optimally addressed through insurance strategies, as opposed to being inefficiently addressed through an unconditional decrease of the equity allocation.

Our analysis suggests that asset allocation and portfolio construction decisions are intimately related to risk management.

The quintessence of investment management is essentially about finding optimal ways to spend risk budgets that investors are reluctantly willing to set, with a focus on allowing for the highest possible access to performance potential while respecting such risk budgets. Risk diversification, risk hedging and risk insurance are three useful approaches to optimal spending of investors’ risk budgets. In this context, improved forms of investment solutions rely on a sophisticated exploitation of the benefits of the three competing approaches to risk management, namely risk diversification (key ingredient in the design of better benchmarks for performance-seeking portfolios), risk hedging (key ingredient in the design of better benchmarks for hedging portfolios) and risk insurance (key ingredient in the design of better dynamic asset allocation benchmarks for long-term investors facing short-term constraints).

In the end, risk management, which focuses on maximising the probability of achieving investors’ long-term objectives while respecting the short-term constraints they face, appears to be the key source of added value in investment management.

The results we obtain confirm that dynamic asset allocation benchmarks can be designed so as to allow for a more efficient spending of investors’ risk budgets. Intuitively, this is because the pre-commitment to reduce the allocation to equity in times and market conditions that require such a reduction so as to avoid over-spending risk budgets, allows investors to invest on average more in equities compared to a simple static strategy that is calibrated so as to respect the same risk budget constraints. The welfare gains involved in this higher allocation to equities are found to be substantial for reasonable parameter values, especially for long-term horizons and in the presence of a mean-reverting equity risk premium.

As a numerical illustration of the benefits of risk-controlled strategies, we first simulate the performance of unconstrained strategies, taking the time-horizon to be equal to 20 years, while the risk aversion parameter, which is not observable, is calibrated in such a way that the average allocation to equity over the 20-year life of the strategy is equal to a target of 10%, 20% or 30%. The three corresponding long-term unconstrained strategies will be referred to as defensive (leading to an average stock weight of 10%), moderate (leading to an average stock weight of 20%) and aggressive (leading to an average stock weight of 30%), respectively.

Exhibit 1 shows the resulting distribution of unconstrained terminal wealth for various risk aversion levels. We find the usual risk-return trade-off: strategies implemented by less risk-averse investors will contain a higher allocation to equities, which will result in a higher average wealth level as well as a higher uncertainty around the terminal wealth level.
Executive Summary

Exhibit 1: Distributions of Terminal Wealth Generated by Long-Term Investment Strategies

Exhibit 2: Risk and Performance Indicators for Long-Term Investment Strategies

<table>
<thead>
<tr>
<th></th>
<th>Aggressive</th>
<th>Moderate</th>
<th>Defensive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min Wealth</td>
<td>251.71</td>
<td>280.10</td>
<td>293.75</td>
</tr>
<tr>
<td>Q5</td>
<td>362.39</td>
<td>345.16</td>
<td>324.78</td>
</tr>
<tr>
<td>Low Target Wealth (Q25)</td>
<td>419.64</td>
<td>378.00</td>
<td>338.89</td>
</tr>
<tr>
<td>Medium Target Wealth (Q50)</td>
<td>459.49</td>
<td>399.71</td>
<td>348.11</td>
</tr>
<tr>
<td>High Target Wealth (Q75)</td>
<td>500.33</td>
<td>421.75</td>
<td>357.40</td>
</tr>
<tr>
<td>Q95</td>
<td>567.38</td>
<td>456.22</td>
<td>371.75</td>
</tr>
<tr>
<td>Max Wealth</td>
<td>739.16</td>
<td>540.35</td>
<td>403.29</td>
</tr>
<tr>
<td>Average Wealth</td>
<td>461.37</td>
<td>400.15</td>
<td>348.18</td>
</tr>
<tr>
<td>High minus Low</td>
<td>80.69</td>
<td>43.75</td>
<td>18.51</td>
</tr>
<tr>
<td>(High minus Low)/(2 × Medium)</td>
<td>8.78%</td>
<td>5.47%</td>
<td>2.66%</td>
</tr>
<tr>
<td>Max 3Y-loss</td>
<td>15.89%</td>
<td>9.26%</td>
<td>7.75%</td>
</tr>
<tr>
<td>Max DD</td>
<td>24.40%</td>
<td>17.78%</td>
<td>15.16%</td>
</tr>
</tbody>
</table>

While long-term strategies are engineered to achieve optimal risk/return trade-offs over the long term, short-term losses and drawdown levels can remain extremely large, especially for the aggressive investor, with a maximum drawdown at 24.4%, as can be seen from Exhibit 2.

In this context, an investor wishing to (or obliged to) maintain the maximum drawdown around say 15% would have to opt for the defensive strategy, even if the level of wealth achieved with this strategy is much less attractive than what is allowed by the aggressive strategy. In this context, the objective measure of the opportunity cost associated with a 15% max drawdown can be formally defined as the additional initial contribution needed by the defensive strategy to reach the same average wealth level as with the aggressive strategy – a cost which turns out to be a prohibitive 32.51% in this particular example.

A less costly solution is to use insurance, as opposed to hedging, to manage downside risk. Hence, as an alternative to opting for the defensive strategy, the investor can
choose the aggressive strategy, which allows for a much higher access to the equity risk premium, and for the implementation of a dynamic risk-controlled investing overlay designed to ensure that the maximum drawdown will be kept below 15% (see Exhibit 3 for the resulting distribution of terminal wealth, where we have also tested maximum drawdown levels at 10% and 20%).

We now observe from Exhibit 3 that the average wealth of the aggressive strategy with a 15% maximum drawdown constraint is substantially higher than the unconstrained defensive one, for essentially the same level of extreme losses. This result makes a strong case for the management of short-term constraints through dynamic risk budgeting rather than through the choice of unnecessarily conservative investment policies. So as to provide an objective assessment of the opportunity cost of imposing stricter drawdown constraints, when these constraints are optimally managed through insurance techniques, we find that the aggressive benchmark needs a mere 5.38% of additional investment with a maximum drawdown constraint of 15% to reach the same average wealth level as with the aggressive strategy without maximum drawdown constraints. This value compares very favourably to the afore-mentioned 32.51% opportunity cost involved in managing maximum drawdown constraints inefficiently through an excessive level of hedging.

Overall, these results illustrate that not disentangling long-term risk-aversion and short-term loss-aversion may lead to poor investment decisions. Relatively simple solutions exist that can be implemented as dynamic asset allocation strategies in order to control short-term risk levels while maintaining an access to long-term sources of performance. These solutions are a substantial improvement with respect to traditional strategies without dynamic risk-control, which inevitably lead to under-spending of investors’ risk budgets in normal market conditions, with a strong associated opportunity cost, and over-spending of investors’ risk budgets in extreme market conditions.
1. Introduction
1. Introduction

The development by Merton (1969) and Merton (1971) of dynamic portfolio theory in the late sixties and early seventies, following pioneering work by Hakansson (1969), Hakansson (1971) and Samuelson (1969), has led to a number of fundamental new insights with respect to simple static portfolio selection models à la Markowitz (1952). In early seminal work, Merton (1971) has shown that the presence of a stochastic opportunity set leads non-myopic long-term investors to incorporate intertemporal hedging demands in addition to the standard speculative motive. Subsequent papers have shown that when maximising utility from terminal (nominal) wealth, an investor only hedges those state variables that impact the nominal short-term rate and the market prices of risk (see for example Detemple et al. (2003)). Some papers have explicitly solved the portfolio choice problem when only one of the two state variables is stochastic, with a separate analysis of the impact of interest rate risk (see for example Lioui and Poncet (2001) and Munk and Sorensen (2004)) and the impact of a mean-reverting equity risk premium (see Kim and Omberg (1996) for the incomplete market case with utility from terminal wealth only, and Wachter (2002) for the complete market case with utility from intermediate consumption). More realistic continuous-time models have subsequently been introduced to simultaneously account for the presence of uncertainty on both interest rates and market prices of risk, either by numerically solving the Hamilton-Jacobi-Bellman equation obtained through dynamic programming (see Brennan et al. (1997)), or, more recently, by exploiting the affine structure of the model (in the sense of Dai and Singleton (2000) and Liu (2007)) and solving the HJB equation explicitly (e.g., Munk et al. (2004) and Sangvinatsos and Wachter (2005)).

One of the main qualitative findings perhaps that stands out from the literature on long-term portfolio decisions with return predictability is the fact that equities serve as a hedge against unfavorable equity returns in the presence of mean-reverting equity returns.1 As a result, the optimal allocation to stocks is higher compared to the myopic case, and investors with longer time-horizons hold more stocks than those with shorter horizons (Kim and Omberg (1996) and Wachter (2002)).2 This prescription has very often been taken at face value by a number of long-term institutional investors such as pension funds and sovereign wealth funds, who explicitly use this argument to justify a high allocation to equity. It is also becoming more and more popular amongst private investors, who can now access, through target date funds or life-cycle funds, an investment solution advocating a deterministic decrease of equity allocations (also known as glidepath) when approaching retirement date. One key problem, however, is that this prescription can lead to extremely difficult situations when risk is assessed from a shorter-term perspective, in particular in the context of a severe bear equity market such as the one experienced in 2008. In fact, most, if not all, investors, even those with the longest possible horizons (such as pension funds or sovereign wealth funds) inevitably face a number of short-term performance constraints, imposed by accounting and/or regulatory pressure, political pressure,

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1 - Empirical evidence for the existence of mean reversion in equity returns have first been provided for the US markets by Poterba and Summers (1988) and Fama and French (1988) (see also Hatziathanassides and Mesomiles (2005) for further evidence on G7 countries).

2 - The presence of parameter uncertainty (Pastor and Stambaugh, 2009) or the desire to hedge against income risk (Benzoni et al., 2007) would have an impact on the term structure of equity allocation.
1. Introduction

peer pressure, etc. In a private wealth management context, there is also strong evidence that investors typically face (mostly self-imposed) short-term constraints, e.g., maximum drawdown constraints.

In other words, it appears that one key element missing from the literature on life-cycle investing is how the presence of short-term performance constraints faced by investors would affect the optimal long-term allocation decisions. The problem of portfolio selection with performance constraints has in fact been examined by a totally separate strand of the literature, which deals with portfolio choice with minimum performance constraints. This literature has shown that the presence of such constraints leads long-term investors to adjust the allocation to risky versus risk-free assets in a state-dependent manner as a function of a suitably-defined risk budget state variable. It starts with a series of papers on portfolio insurance, which focus on strategies aiming to guarantee a given minimum level of terminal wealth, based either on Constant-Proportion Portfolio Insurance (CPPI) introduced by Black and Jones (1987) and Black and Perold (1992), or on Option-Based Portfolio Insurance (OBPI) strategies (see Leland (1980)). More recently, Grossman and Zhou (1996) show that OBPI strategies are optimal for investors who maximise their expected utility from terminal wealth subject to the constraint that this wealth should be greater than or equal to some fixed floor. In a related effort, Basak (2002) suggests introducing an implicit rather than such an explicit constraint on terminal wealth, which amounts to assuming that marginal utility goes smoothly to infinity as wealth shrinks to the floor. Several papers have generalised these models by imposing minimum performance constraints involving the choice of a stochastic, as opposed to deterministic, benchmark. Teplá (2001) shows that the optimal strategy in the presence of such constraints involves a long position in an exchange option. Basak et al. (2006) generalise the approach by considering less stringent constraints on performance, i.e., by imposing that the minimum performance should be achieved with a given probability of 95%, 99%, etc., i.e. a probability of less than 1. In all these studies, investment opportunities are constant so that optimal strategies in the absence of performance constraints are of the fixed-mix nature, and the presence of return predictability is simply not accounted for. An extension to the case of stochastic opportunity set has been proposed by El Karoui et al. (2005), who show that OBPI maximises expected utility in a more general framework where risk and return parameters are not necessarily constant. However, they adopt a reduced-form approach, directly modelling the unconstrained strategy. In particular, the impact of stock return predictability on the optimal unconstrained strategy does not appear, while it is at the heart of the conflict between long-term objectives and short-term constraints.

The purpose of this paper is to tie together these two separate strands of the literature by extending Grossman and Zhou (1996), Basak (2002) and Teplá (2001) to the case where investment opportunities are stochastic. In contrast with El Karoui et al. (2005), who take the underlying
strategy of the OBPI to be exogenously given, we derive it by maximising utility in a long-term investment model. Symmetrically, our paper can be seen as an attempt to extend the literature on long-term investing by analysing how the presence of short-term constraints impacts the optimal long-term asset allocation decisions. More specifically, we solve for the optimal allocation of a long-term investor facing short-term constraints in the presence of both interest rate and unspanned risk premium uncertainty, by using the martingale approach to portfolio optimisation in incomplete market settings. The performance constraints that we consider are expressed under a general form that nests both deterministic performance constraints of portfolio insurance and stochastic performance constraints that can be related to the presence of an asset-driven benchmark as in Teplá (2001) or a liability-driven benchmark as in Martellini and Milhau (2009). Our main result is an explicit representation of the relationship between optimal strategies in the presence and in the absence of short-term constraints, expressed both in terms of optimal payoffs and optimal portfolio weights. This result, which to the best of our knowledge is new to this literature, allows us to disentangle the impact of short-term constraints from the impact of return predictability on the optimal allocation decision. An attempt to reconcile these two strands of the literature appears all the more useful that the two motivations behind dynamic asset allocation decisions, namely the risk management and the tactical motivations, are often perceived as inconsistent and mutually exclusive.

Hence, dynamic risk-controlled strategies, which typically imply a reduction to equity allocation when a drop of equity prices has led to a substantial diminution of the risk budget, have often been blamed for their pro-cyclical nature, and long-term investors are often reluctant to sell equity holdings in those states of the world where equity markets have become particularly attractive given the presence of mean-reversion in the equity risk premium. In other words, it is widely perceived that a tension exists between a focus on hedging long-term risk and a focus on insurance with respect to short-term constraints. We cast new light on this debate by showing that long-term objectives and short-term constraints need not be mutually exclusive, and that they can naturally coexist within the context of a long-term investing strategy consistent with short-term performance constraints. In an empirical analysis, we show that depending on parameter values the risk-controlled motivation may outweigh the tactical motivation, or vice-versa, for a given time and state of the world. We also show that the presence of upper (in addition to lower) bounds on performance, consistent with the kind of utility satiation often exhibited by long-term investors, is another, independent reason why a fall in equity prices should not always lead to a decrease of equity allocation, regardless of the presence of mean-reversion in the equity risk premium.

The rest of the paper is organised as follows. In Section 2, we solve for the optimal investment strategy for an investor who faces uncertainty in interest rates and the equity risk premium. In
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Section 3, we impose short-term minimum performance constraints and derive the new optimal portfolio rule. In Section 4 we measure the cost of insurance against downside risk and show that it can be reduced by imposing a maximum performance constraint. Section 5 gives several examples of floors and optimal constrained strategies. Section 6 provides an empirical illustration of the benefits of using the optimal strategy when exogenous short-term constraints are imposed. Section 7 presents a similar analysis in the presence of drawdown constraints, with a floor that depends on past values of the portfolio. In Section 7.3, we discuss implementation challenges, and we conclude in Section 8. Proofs of the main results and technical details can be found in a dedicated appendix.
1. Introduction
2. Long-Term Portfolio Choice Over the Life-Cycle
2. Long-Term Portfolio Choice Over the Life-Cycle

In this section we introduce a formal investment model for portfolio decisions over the long run.

2.1 State Variables and Financial Assets

Uncertainty in the economy is modeled through a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\mathcal{F}\) is a sigma-algebra and \(\mathbb{P}\) a probability measure that represents the investor’s beliefs. The finite time span is denoted with \([0, T]\), and the probability space is endowed with a filtration \((\mathcal{F}_t)_{t \in [0,T]}\) that satisfies the usual conditions: \(\mathcal{F}_t\) is the information set to which the investor has access at time \(t\).

The nominal risk-free rate is denoted with \(r\), and follows a mean-reverting process (Vasicek, 1977):

\[
\frac{dr_t}{r_t} = \left\{ a(b - r_t) + \sigma_r \, dz^r_t \right\}.
\]

There exists a locally risk-free asset, whose value is the continuously compounded interest rate:

\[
S^0_t = e^{\int_0^t r_s \, ds}.
\]

We will refer to this asset as the bank account, or cash. Assuming a constant price of interest rate risk \(\lambda_r\), we have that the price of a zero-coupon bond maturing at date \(T\) is:

\[
B(t, T) = \exp \left\{ -D(T-t)r_t + C(T-t) \right\},
\]

where:

\[
D(u) = \frac{1 - e^{-au}}{a},
\]

\[
C(u) = \left( b - \frac{\sigma_r \lambda_r}{a} \right) \left[ \frac{1 - e^{-au}}{a} - u \right]
\]

\[
+ \frac{\sigma_r^2}{2a^2} \left[ u^2 - 2 \frac{1 - e^{-au}}{a} + \frac{1 - e^{-2au}}{2a} \right].
\]

All zero-coupon bonds can be replicated by dynamically trading in the bank account and in a zero-coupon bond with constant maturity \(\tau_B\), whose price evolves as:

\[
\frac{dB_t}{B_t} = \left\{ r_t - D(\tau_B)\sigma_r \lambda_r \right\} dt - D(\tau_B) \sigma_r \, dz^r_t.
\]

(2.2)

Note that for the risk premium on long-term bonds over short-term bonds to be positive, the parameter \(\lambda_r\) must be negative.

In addition to the bank account and the constant-maturity bond, the investor also has access to a stock index \(S\), that evolves as:

\[
\frac{dS_t}{S_t} = \left\{ r_t + \sigma_S \lambda_S \right\} dt + \sigma_S \, dz^S_t.
\]

The (instantaneous) Sharpe ratio of the stock is assumed to follow a mean-reverting process:

\[
\frac{d\lambda^S_t}{\lambda^S_t} = \kappa \left( \lambda - \lambda^S \right) \, dt + \sigma_\lambda \, dz^\lambda_t.
\]

This model was introduced by Kim and Omberg (1996) to account for predictability in excess returns on stocks. The mean-reversion combined with a negative correlation of \(z^S\) and \(z^\lambda\) generates a decreasing term structure of volatility in excess returns (see e.g. Martellini and Milhau (2011a)).

We emphasise that the volatility of the stock in our model is constant. This assumption is obviously at odds with the abundant empirical evidence that the volatility of stock returns is stochastic. Stochastic volatility in itself is not an issue in optimal portfolio choice: it generates a
hedging demand only if it has an impact on Sharpe ratios. Deguest et al. (2012) show that in the presence of stochastic volatility and a mean-reverting Sharpe ratio, the utility-maximising strategy has the same structure as if volatility was constant, and investors will continuously update the volatility of the stock index in the formula as market conditions change. In particular, they will reduce their holdings in stocks in response to a rise in volatility. Changes in the opportunity set are summarised by changes in the Sharpe ratio and in the interest rate, so investors will not react to changes in volatility unless they translate into changes in either one of these two state variables.

There are three stochastic processes in our model: $S$, $r$, and $\lambda^S$. We define $V$ to be the state vector

$$V_t = \left( \log S_t \quad r_t \quad \lambda^S_t \right)'.$$ (2.3)

The logarithm of the stock price is used instead of the price itself so as to have a Gaussian vector, a property that is useful for the calibration of the model.

Uncertainty in the model can be summarised by a 3-dimensional Brownian motion $z$. The loadings of innovations to the state variables on this Brownian motion are gathered in the volatility vectors, that verify, by definition:

$$\sigma'_t dz_t = \sigma_t dz^z_t, \quad \text{for } i = r, S, \lambda.$$ (2.4)

Therefore (see Harrison and Kreps (1979)), the augmented filtration generated by $V$ coincides with the augmented filtration generated by $z$. This filtration is the information to which the investor has access at date $t$: it consists of all current and past values of processes $S$, $r$, and $\lambda^S$.

The vector $V$ has a volatility matrix $\sigma_V$, which can be written as:

$$\sigma_V = \begin{pmatrix} \sigma_S & \sigma_r & \sigma_\lambda \end{pmatrix}.$$

We let $\rho_{ij}$ denote the correlation between state variables $i$ and $j$, for $i, j = r, S, \lambda$, so that $\rho_{ij} = \frac{\sigma_i \sigma_j}{\sigma_i \sigma_j}$. The volatility vector of the bond (2.2) can be obtained as:

$$\sigma_B(\tau_B) = -D(\tau_B)\sigma_r.$$

The volatility matrix of the traded assets is obtained by concatenating horizontally the volatility vectors of the bond and the stock:

$$\sigma = \begin{pmatrix} \sigma_B(\tau_B) & \sigma_S \end{pmatrix}.$$

### 2.2 State-Price Deflators

By definition, a market price of risk is any vector process $\lambda$ such that the product $\sigma' \lambda$ equals the vector of expected excess returns on assets. Among those processes, the one that is in the span of the volatility matrix is of particular interest, and will be denoted as $\lambda$:

$$\lambda_t = \sigma(\sigma')^{-1} \begin{pmatrix} -D(\tau_B)\sigma_r \lambda_r \\ \sigma_S \lambda^S_t \end{pmatrix},$$ (2.5)

which can be decomposed as $\lambda_t = \Lambda_1 + \lambda^S_t \Lambda_2$, with:

$$\Lambda_1 = \sigma(\sigma')^{-1} \begin{pmatrix} -D(\tau_B)\sigma_r \lambda_r \\ 0 \end{pmatrix},$$

$$\Lambda_2 = \sigma(\sigma')^{-1} \begin{pmatrix} 0 \\ \sigma_S \end{pmatrix}.$$

As shown by He and Pearson (1991), the other prices of risk are those processes of the form $\lambda + \nu$, where $\sigma' \nu_t = 0$ almost surely for all $t$. To each of these prices of risk is associated a state-price deflator, defined as:
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\[ M_t = \exp \left[ -\int_0^t \left( r_s + \frac{\| \lambda_s \|^2 + \| \nu_s \|^2}{2} \right) ds - \int_0^t (\lambda_s + \nu_s)' d\pi_s \right], \quad t \geq 0. \tag{2.6} \]

(We do not explicitly indicate the dependence of \( M \) upon \( \nu \) if there is no risk of confusion). In general there are infinitely many prices of risk because the matrix \( \sigma \) does not have full row rank: it has two columns and three rows, due to the presence of a specific risk factor in the Sharpe ratio that is not spanned by bonds and stocks. However, there is an important special case in which the price of risk is unique: it is the case where \( \rho \lambda \lambda = -1 \) (an assumption made in Wachter (2002)). Then there are only two independent sources of risk in the model, so the Brownian motion \( z \) can be defined in 2 dimensions, and the matrix \( \sigma \) becomes square and invertible. In this case the market is dynamically complete, since all risk factors are spanned by bonds and stocks.

2.3 Optimal Portfolio Choice
The investor has a finite horizon \( T \), which can be thought of as retirement date. Over the period \([0, T]\), he can continuously rebalance his portfolio between bonds, stocks and cash. If we let \( w_t \) be the vector of weights allocated to bonds and stocks at date \( t \), the budget constraint can be written as:

\[ \frac{dA_t}{A_t} = \left[ r_t + w'_t \sigma' \lambda_t \right] dt + w'_t \sigma' d\pi_t. \tag{2.7} \]

The investor is primarily concerned with the distribution of his terminal wealth \( A_T \), because it is the wealth available just before retirement. His preferences for high return and low risk are represented by the expected utility from \( A_T \), where the utility function considered in this paper is the Constant Relative Risk Aversion (CRRA) one:

\[ U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x > 0. \]

According to these preferences, the objective of the investor is to achieve the highest expected utility:

\[ \max_{w} \mathbb{E} [U(A_T)] \text{ subject to (2.7)].} \tag{2.8} \]

This dynamic portfolio choice problem can be solved by dynamic programming techniques (Merton, 1973). It can also be solved via the duality technique introduced by Cox and Huang (1989), extended to incomplete markets by He and Pearson (1991). We follow the latter approach here. First, the optimal terminal wealth is derived. Its expression involves the \textit{minimax} pricing kernel, which is obtained by using a special value of \( \nu \), denoted as \( \nu^* \), in (2.6). Second, the optimal wealth is priced and the replicating portfolio is obtained. The replicating portfolio of the optimal terminal wealth is the solution to the original problem (2.8). The following proposition provides the solution to this problem. This solution is quasi-analytical in that the optimal weights are expressed in terms of functions of time-to-horizon that can be obtained by solving a system of ordinary differential equations (ODEs).

\textbf{Proposition 1} Let \( M^\nu \) denote the minimax pricing kernel in the sense of He and Pearson (1991) in (2.8). Then:
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- We have:

\[ M^*_{t} = \exp \left[ -\int_{0}^{t} r_s \, ds - \frac{1}{2} \int_{0}^{t} (\| \lambda_s \|^2 + \| \nu^*_s \|^2) \, ds \right. \]

\[ \left. - \int_{0}^{t} (\lambda_s + \nu^*_s)' \, dz_s \right], \]

where:

\[ \nu^*_t = -(1 - \gamma) \left[ C_2(T - t) + C_3(T - t) \lambda^S \right] N \sigma \lambda. \]

- The optimal wealth process is:

\[ A^*_t = \frac{A_0}{G_0 B(0, T)^{-1/2}} (M^*_t)^{-1/2} B(t, T)^{1/2} G_t, \]

\[ G_t = \exp \left[ \frac{1 - \gamma}{\gamma} \left[ C_1(T - t) + C_2(T - t) \lambda^S \right] \lambda^S \lambda \right], \]

where:

\[ w^*_t = \frac{\lambda^M_t}{\gamma \tilde{c}_t} \lambda^M_t + \left(1 - \frac{1}{\gamma} \right) \beta^B_t \lambda^B, \]

\[ - \left(1 - \frac{1}{\gamma} \right) \left[ C_2(T - t) + C_3(T - t) \lambda^S \right] \lambda^S \lambda, \]

The functions \( C_1, C_2 \) and \( C_3 \) are solutions to a system of ODEs written in Appendix A.1. In these equations, \( N = I_3 - \sigma (\sigma' \sigma)^{-1} \sigma \) is the matrix of the residual of the orthogonal projection on the columns of \( \sigma \).

- The optimal strategy reads:

\[ w^M_t = \frac{1}{1' (\sigma' \sigma)^{-1} \sigma' \lambda_t} (\sigma' \sigma)^{-1} \sigma' \lambda_t, \quad w_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

\[ w^B_t = \frac{1}{1' (\sigma' \sigma)^{-1} \sigma' \sigma \lambda} (\sigma' \sigma)^{-1} \sigma' \sigma \lambda, \]

\[ \frac{\lambda^M_t}{\lambda^M_t} = 1' (\sigma' \sigma)^{-1} \sigma' \lambda_t, \quad \beta^B_t = \frac{D(T - t)}{D(\tau_B)}, \]

\[ \beta^B_t = 1' (\sigma' \sigma)^{-1} \sigma' \sigma \lambda. \]

Proof. See Appendix A.1.

It can be directly checked that \( M^*_t \) is a pricing kernel, by showing that the random variable

\[ Z_T = \exp \left[ -\frac{1}{2} \int_{0}^{T} (\| \lambda_t + \nu^*_t \|^2) \, ds \right. \]

\[ \left. - \int_{0}^{T} T (\lambda_t + \nu^*_t)' \, dz_t \right] \]

has expectation equal to 1, and thus defines a change of probability measure. To show that \( \mathbb{E}[Z_T] = 1 \), it suffices to show that the following technical condition is satisfied:

\[ \sup_{x \in [0, T]} \mathbb{E} \left[ \exp \left( x \| \lambda_t + \nu^*_t \|^2 \right) \right] < \infty \]  

(2.11)

for some positive \( x \).

This condition indeed implies that the Novikov criterion is satisfied, and the Novikov criterion implies in turn that \( \mathbb{E}[Z_T] = 1 \) (see Karatzas and Shreve (2000), Corollary 5.13). To show that (2.11) holds, we must essentially show that the supremum of \( \mathbb{E} \left[ \exp \left( x \| \lambda_t + \nu^*_t \|^2 \right) \right] \), for \( t \) in \([0, T]\) and a sufficiently small \( x \), is finite. This can be achieved by directly computing this expectation, which poses no problem because \( \lambda^S \) is normally distributed, and by showing that it is a continuous function of time over the compact interval \([0, T]\). This continuity property ensures that the supremum is finite, hence that (2.11) holds, hence that \( Z_T \) indeed defines a change of probability measure.

The decomposition (2.10) is a special case of the general decomposition result established by Detemple and Rindisbacher (2010): the optimal portfolio is represented as a combination of three building blocks. The first block is the mean-variance
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portfolio, which maximises the Sharpe ratio over the next trading period:

$$w_t^{MV} = \arg \max_{w_t} \frac{w_t' \sigma' \lambda_t}{\sqrt{w_t' \sigma' \sigma w_t}}.$$

The allocation to the mean-variance component in the optimal strategy is increasing in the Sharpe ratio $\lambda_t^{MV}$, and decreasing in the volatility $\sigma_t^{MV}$.

The second block is a portfolio entirely invested in nominal bonds. We note that the strategy $\beta B^{\beta} w_{\beta}$, which is invested in the constant-maturity bond and in cash, is the strategy that replicates a zero-coupon of maturity $T$. The second term on the right side of (2.10) therefore represents a demand for a zero-coupon of maturity equal to the investor’s horizon. This demand is increasing in the risk aversion.

The third block is a portfolio that hedges unexpected changes in the Sharpe ratio of the stock. Formally, $w_\lambda$ is the portfolio that maximises the squared correlation with the innovations to the Sharpe ratio:

$$w_\lambda = \arg \max_{w_t} \frac{(w_t' \sigma' \lambda)^2}{\sigma_\lambda^2 \| \sigma w_t \|^2}, \text{ s.t. } w_t' 1 = 1.$$

The allocation to this block depends on the functions $C_2$ and $C_3$, which can be obtained by numerically solving the ODEs written in the proposition, and it depends on the current Sharpe ratio of the stock. It is easier to interpret the impact of this hedging demand on the allocation if one assumes $\rho_{\lambda S} = -1$. Then $w_\lambda$ is entirely invested in stocks, and the beta is the negative of the ratio of volatilities $\sigma_\lambda / \sigma_S$:

$$\beta_\lambda w_\lambda = -\frac{\sigma_\lambda}{\sigma_S} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence the investor allocates more to stocks than if the Sharpe ratio was constant. The increase in allocation is determined by the functions $C_2$ and $C_3$, which are increasing in the time-to-horizon: hence young investors (with a large $T - t$) will respond more to predictability in stock returns.

We conclude this section by giving an analytical expression for the indirect utility, which is the expected utility from $A_T^{nu}$. Appendix A.1 shows that:

$$\mathbb{E}_t [ U (A_T^{nu}) ] = U \left( \frac{A_T^{nu}}{B(t, T)} \right) G_t^\gamma.$$

This expression will be used to compare the expected utility of strategies that incorporate floor constraints to that of the unconstrained strategy.
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The objective (2.8) is fundamentally a long-term one, since it is defined in terms of the terminal wealth $A_T$. In practice, investors may be subject to short-term performance constraints, stating that their wealth must not fall below a floor equal to $F$. This series of short-term constraints can be formally written as:

$$A_t \geq F_t, \text{ for all } t \leq T. \quad (3.1)$$

(As usual, the inequality is assumed to hold with a probability of 1).

3.1 Floor Process

In this paper, we will focus only on replicable floors. This assumption is made because it ensures that if $A_0 \geq F_0$, then there exists at least one strategy that respects the constraint (3.1): it suffices to invest only in the floor-replicating portfolio (henceforth, FRP), which generates the wealth $A_t = \frac{dA_t}{dF_t}$, a quantity which is greater than $F_t$. Hence, without loss of generality, we can assume that the floor is the wealth generated by some dynamic trading strategy in nominal bonds and stocks:

$$\frac{dF_t}{F_t} = \left[ r_t + (w^F_t)' \sigma \lambda_t \right] dt + (w^F_t)' \sigma dz_t. \quad (3.2)$$

A minimum information requirement is that the process $w^F$ be progressively measurable with respect to the filtration $\mathcal{F}$. Since we have seen (see Section 2) that this filtration is generated by the state vector $V$, imposing this restriction is equivalent to requiring that $w^F$ be a function of time $t$ and current and past values of $V$:

$$w^F_s = w^F(t, V_u; s \leq t). \quad (3.3)$$

Of course, if (3.1) is satisfied, then we have $A_T \geq F_T$ with probability 1. While (3.4) seems to be much less restrictive than (3.1), the converse implication also holds true, because the payoff $F_T$ is replicable. Indeed, by absence of arbitrage opportunities, if a portfolio yields an almost surely nonnegative payoff, it must be the case that at all previous dates, the value of the portfolio is nonnegative as well. Hence the series of short-term constraints (3.1) is equivalent to the single long-term constraint (3.4). Nevertheless, a long-term performance constraint such as (3.4) implies a short-termist behaviour, in that the investor wants his wealth to be above the floor at all instants. If wealth were to fall below the floor at some date $t$, it would be impossible to respect the terminal constraint (3.4).

3.2 Optimal Terminal Wealth Under Performance Constraints

As pointed out above, it is possible to satisfy the constraints (3.1) by investing only in the FRP. This strategy is, however, very conservative, and it is unlikely to be optimal if the investor has some appetite for risk. This is why we consider the following, more general, portfolio choice problem:

$$\max \mathbb{E} [U(A_T)], \text{ subject to short-term constraints (3.1) and budget constraint (2.7)],} \quad (3.5)$$

an objective that is equivalent to:

$$\max \mathbb{E} [U(A_T)], \text{ subject to } A_T \geq F_T \text{ and budget constraint (2.7).} \quad (3.6)$$

A somewhat heuristic mathematical argument, employed in Martellini and Milhau (2009), suggests that Option-Based Portfolio Insurance (OBPI) is optimal.
Consider indeed the static program obtained with the unconstrained minimax pricing kernel:

\[
\max_X \mathbb{E}[U(X)],
\]

subject to \(X \geq F_T\)

and \(\mathbb{E}[M_T X] = A_0\).

Its "Lagrangian" is:

\[
\mathcal{L} = U(X) - \eta_1 [M_T X - A_0] + \eta_2 (X - F_T),
\]

and the Karush-Kühn-Tucker conditions at the optimum \(X^*\) read:

\[
(X^*)^{-\gamma} - \eta_1 M_T X^* + \eta_2 = 0, \\
\eta_2 \geq 0, \quad \eta_2 (X^* - F_T) = 0, \quad X^* \geq F_T, \\
\mathbb{E}[M_T X^*] = A_0.
\]

These conditions imply that:

\[
X^* = \max \left((\eta_1 M_T)^{-\frac{1}{2}}, \ F_T \right),
\]

a payoff that can be rewritten as:

\[
X^* \overset{\text{def}}{=} F_T + (\xi M_T - F_T)^+, \quad (3.7)
\]

where \(\xi\) is a constant. This constant is implicitly defined by the budget constraint \(\mathbb{E}[M_T X] = A_0\). The following result shows that \(X^*\) is indeed utility-maximising, a result that is present in Grossman and Zhou (1996) and El Karoui et al. (2005).

**Proposition 2** Let \(A_T\) be any terminal wealth that satisfies the constraint (3.4). Then \(X^*\) dominates \(A_T\) in the sense of expected utility.

\[
\mathbb{E}[U(A_T)] \leq \mathbb{E}[U(X^*)].
\]

**Proof.** See Appendix A.2.

Hence, if the payoff \(X^*\) is replicable, it is the optimal terminal wealth in (3.6). It remains to examine under which conditions \(X^*\) is replicable. We already know that \(A_T^{wu}\) is replicable (by construction) and that \(F_T\) is replicable too (by assumption). But these two conditions are not sufficient to ensure that the exchange option in (3.7) is replicable. To understand why, it suffices to think of the case of a European call written on an underlying asset with stochastic volatility. The payoff of the call may not be attainable, even if the underlying is traded, because volatility risk is a potential source of incompleteness. In general, the option will be replicable only if volatility risk is spanned, either by the stock itself (for instance, if there is a perfect negative correlation between volatility risk and stock price risk) or by a tradable contract written on volatility (for instance, either a futures contract on volatility or a derivative contract whose value depends on volatility).

The situation is similar here. Even if the two underlying assets of the exchange option are replicable, there is no guarantee that the option is replicable. Sufficient conditions for replicability include restrictions on the volatility of the ratio \(R^{wu} \overset{\text{def}}{=} A^{wu}/F\). Ito's lemma shows that the volatility vector of \(R^{wu}\) is:

\[
\sigma_t^{R^{wu}} = \sigma [w_t^{wu} - w_t^F],
\]

which, using (2.10), can be rewritten as:

\[
\sigma_t^{R^{wu}} = \frac{1}{\gamma} \lambda_t^u + \left(1 - \frac{1}{\gamma}\right) \sigma_B (T-t) \]

\[
- \left(1 - \frac{1}{\gamma}\right) C_2 (T-t) \sigma (\sigma')^{-1} \sigma'_u - \sigma w_t^F. \quad (3.8)
\]

Hence \(\sigma^{R^{wu}}\), the scalar volatility of \(R^{wu}\), depends on the current value of the Sharpe ratio through the volatility of \(A^{wu}\). It can also depend on all past and current values of \(V\) through the portfolio weights \(w_t^F\).
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(see (3.3)). Hence, for the payoff to be replicable, it suffices to assume that the sources of risk that impact \( V \) are spanned, which leads to the following proposition.

**Proposition 3** Assume that \( \sigma^{R_u} \) is progressively measurable with respect to \( \mathcal{F} \), the filtration generated by \( V = (\log S, r, \lambda^S)' \), and that the three sources of risk \( z^S, z^r \) and \( z^\lambda \) are spanned. Then the payoff \( X^* \) defined in (3.7) is replicable, and is therefore the optimal terminal wealth in (3.6).

Three comments are in order. First, in our model, if the three sources of risk are spanned, then the market is complete. But it is important to stress that market completeness is a sufficient condition, not a necessary one. It is possible to relax this assumption by imposing that only those state variables that impact the volatility of \( R_u \) be spanned. For instance, if the FRP is such that \( \sigma^{R_u} \) does not actually depend on \( \lambda^S \), then \( z^\lambda \) needs not be spanned for \( X^* \) to be replicable.

Second, in the framework of Section 2, assuming a complete market is equivalent to assuming that the equity premium risk is spanned. This assumption is in fact not unrealistic. Empirical calibrations of the stochastic Sharpe ratio model of Section 2 report estimated values for the correlation \( \rho_{S\lambda} \) that are very close to \(-1\). Values comprised between \(-1\) and \(-90\%\) can be found in Campbell and Viceira (1999), Barberis (2000) and Xia (2001), and the author’s own calibration on US data yields a value of \(-94.94\%\). The assumption of a perfectly negative correlation has been made by Wachter (2002) and Munk et al. (2004).

Third, if one does not assume that the state variables that impact the volatility of \( R_u \) are spanned, there is no guarantee that the payoff \( X^* \) is attainable. In such a situation, it is much more difficult to guess the optimal terminal wealth in the constrained problem (3.6). An alternative approach is to try to approximate “at best” the payoff \( X^* \). There are numerous possible replication criteria, like the minimization of quadratic replication error (see e.g. Schweizer (2001) for a survey of these approaches). Unfortunately, the payoff that achieves the best approximation of \( X^* \) according to such criteria is not necessarily the one that leads to the highest expected utility.

### 3.3 Optimal Dynamic Allocation Strategy Under Performance Constraints

We now focus on the optimal portfolio strategy in program (3.6). It follows from the previous discussion that if the equity premium risk is spanned (and thus, the market is complete), the optimal strategy is the strategy that replicates the terminal wealth

\[
A_T^* = F_T + [\xi A_T^{su} - F_T]^+.
\]

Using the put-call parity relationship, we can rewrite this as:

\[
A_T^{*C} = \xi A_T^{su} + [F_T - \xi A_T^{su}]^+.
\]

This decomposition shows that the optimal terminal wealth is the sum of the wealth generated by investing \( \xi A_0 \) in the optimal unconstrained strategy, and the remainder of available wealth, \((1 - \xi)A_0\), in an exchange option. This option can be interpreted as an insurance put, since it provides the owner with the possibility of exchanging the payoff \( \xi A_T^{su} \) for the payoff
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$F_T$. As a consequence, in any state of the world, the investor ends up with a wealth greater than or equal to $F_T$.

Since an option payoff appears in the right side, we expect the strategy to involve some Greeks of this option, i.e. some first-order derivatives of the put price

$$P_t = \mathbb{E}_t \left[ \frac{M^u_t}{M^u_0} [F_T - \xi A^u_T]^+] \right].$$

This price can be rewritten as an expectation under $Q^F$, which is the probability measure under which asset prices expressed in the numeraire $F$ follow martingales:

$$\frac{dQ^F}{d\mathbb{P}}|_t = \frac{M^u_t F_t}{F_0}.$$  \hspace{1cm} (3.11)

With this notation, we have:

$$P_t = F_t \mathbb{E}^{Q^F}[ (1 - \xi R^u_T)]^+] .$$

The conditional expectation on the right side is the put price expressed in the numeraire $F$. Because the volatility of $R^w$ is $\mathcal{F}$-adapted, it is a function of $(t, V^s; s \leq t)$, hence it is potentially a function of all past and current values of state variables $S$, $r$ and $\lambda$. In order to have only a finite number of Greeks appearing in the optimal strategy, we must ensure that the put price is a function of a finite number of state variables only. From now on, we thus restrict the volatility of $R^w$ to be a function of the current values $V_t$. This condition is satisfied as soon as the weights of the FRP are themselves functions of these current values. We emphasise that this assumption is not required for the replicability of $X^*$, but is made only in order to ensure that the number of hedging demands appearing in the optimal strategy is finite.

The following proposition summarises the assumptions made so far, and the form of the optimal payoff and strategy.

**Proposition 4** Assume that:
- equity premium risk is spanned (e.g., $\rho_{S\lambda} = -1$);
- the vector $\mathbf{w}_t^F$ is a function of time and current values of stochastic processes $V_t$;
- the initial wealth satisfies $A_0 \geq F_0$.

Then the optimal terminal wealth in (3.6) is $A_T^*$ given in (3.10), where the constant $\xi$ satisfies:

$$\mathbb{E} [M^u_T A_T^*] = A_0.$$

Then the optimal strategy is:

$$\mathbf{w}_t^* = \left( 1 - m_t F_t \right) \mathbf{w}_t^{nu} + m_t F_t A_t^{nu} \mathbf{w}_t^{F} + \frac{F_t}{A_t^{nu}} (\sigma' \sigma)^{-1} \sigma' \sigma V_t p_3, $$

where:

$$m_t = p - \xi R^u_T p_2.$$  \hspace{1cm} (3.13)

$p_2$ and $p_3$ denote the partial derivatives of the put price with respect to its second and third arguments ($p_2$ is a scalar and $p_3$ a $3 \times 1$ vector).

**Proof.** See Appendix A.3.

The optimal strategy (3.13) is expressed in terms of the Greeks of the insurance put and the coefficient $m_t$. It would be of interest to obtain an analytical representation, but this objective is challenging because in general the ratio $R^w$ has stochastic volatility (see (3.8)). It is only under restrictive assumptions that the volatility is constant and a
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closed-form expression for the optimal strategy can be obtained (see for example in Subsection 3.4 below). In other cases, one needs to use option pricing techniques in the presence of stochastic volatility in order to have closed-form expressions for the price and the Greeks (see e.g. Heston (1993)).

The optimal strategy (3.13) involves a "risk budget", defined as the distance of current wealth to a modified floor, which is equal to \( m_t F_t \). This floor is distinct from \( F_t \), unless \( m_t = 1 \). In the following proposition, we summarise several properties of \( m_t \).

**Proposition 5** Other things being equal, the coefficient \( m_t \):
- is comprised between 0 and 1;
- is decreasing in \( A_t^{u*} \);
- tends to one when \( A_t^{u*} \) goes to \( F_t \);
- shrinks to zero when \( A_t^{u*} \) grows to infinity.

**Proof.** See Appendix A.4.

The properties given in Proposition 5 have several implications for the risk budget \( A_t^{u*} - m_t F_t \). First, because \( m_t \) is lower than 1, the risk budget is greater than the distance of wealth to the floor. It might thus seem too high, but it is not because it adjusts in a dynamic way. In particular, it decreases as \( A_t^{u*} \) decreases, and it shrinks to zero when \( A_t^{u*} \) gets closer to the floor \( F_t \), so the allocation to the unconstrained strategy shrinks to zero as well. Last, the relative risk budget \( \frac{A_t^{u*} - m_t F_t}{A_t^{u*}} \) tends to one when \( A_t^{u*} \) becomes infinite. As a consequence, when wealth is large, investors behave as if they were not subject to the performance constraints.

The third property of the risk budget generates a conflict between long-term investment objectives and short-term performance constraints. Indeed, if the unconstrained strategy allocates a large weight to equities, then a downturn in the equity market will cause a sudden decrease in the constrained wealth. The optimal constrained portfolio rule (3.13) then recommends decreasing the allocation to the unconstrained strategy. On the other hand, the mean reversion in equity returns make stocks more attractive after large negative returns, so the unconstrained strategy recommends a higher allocation to equities after this market event. The final effect on the optimal constrained weight allocated to equities is therefore undetermined, and will be the result of the competition between the pro-cyclical decrease in the risk budget and the anti-cyclical increase in the unconstrained weight of stocks. We give an empirical illustration of this effect in Subsection 7.2 below.

3.4 An Analytical Expression When \( A_t^{u*}/F \) Has Deterministic Volatility

In order to have a fully analytical expression for the constrained optimal strategy (3.13), we need a closed-form expression for the put price \( p \). Such an expression is not available in general, because the ratio \( R_t^{u} \) may have stochastic volatility (see (3.8)). For instance, if the FRP is deterministically time-dependent, the volatility is stochastic because the Sharpe ratio of the stock is stochastic. On the other hand, if the FRP is such that the volatility of \( R_t^{u} \) is deterministic, the exchange option can be priced in closed form using the Black-Scholes formula:

\[
p(t, \xi R_t^{u*}, V_t) = N(-d_{2,t}) - N(-d_{1,t}) \xi R_t^{u*},
\]
where:

\[
\begin{align*}
    d_{1,t} &= \frac{1}{\sqrt{\int_t^T (\sigma_s^{R*u})^2 \, ds}} \left[ \ln (\xi R_{t}^{u}) + \frac{1}{2} \int_t^T (\sigma_s^{R*u})^2 \, ds \right], \\
    d_{2,t} &= d_{1,t} - \sqrt{\int_t^T (\sigma_s^{R*u})^2 \, ds},
\end{align*}
\]

\(\sigma^{R*u}\) is the deterministic volatility of \(R^u\), and \(\mathcal{N}\) is the standard normal cumulative distribution function. The option price in the dollar numeraire is thus:

\[
\mathbb{E}_t \left[ \frac{M_T^{R*u}}{M_t^{R*u}} (F_T - \xi A_T^{R*u})^+ \right] = \mathcal{N} (-d_{2,t}) F_t - \mathcal{N} (-d_{1,t}) \xi A_t^{R*u},
\]

which is Margrabe’s formula (Margrabe, 1978). In this setting, we have

\(\nu_R = -\xi \mathcal{N} (-d_{1,t})\), hence

\(\eta_t = \mathcal{N} (-d_{2,t})\)

and the optimal strategy becomes:

\[
\begin{align*}
    w_{t}^{\star} &= \left( 1 - \mathcal{N} (-d_{2,t}) \frac{F_t}{A_t^{R*u}} \right) w_t^{u} \\
    &\quad + \mathcal{N} (-d_{1,t}) \frac{F_t}{A_t^{R*u}} w_t^{F}.
\end{align*}
\]

A simple way to obtain a deterministic volatility of \(R^u\) is to set the volatility of the Sharpe ratio to zero, so as to obtain a deterministically time-dependent \(\lambda^S\), and to choose a deterministic FRP.

### 3.5 Extension to Asset-Liability Management

The objective considered in (3.5) is an “Asset-Management” objective in the sense that investor’s welfare depends exclusively on nominal wealth. For investors endowed with liabilities, this representation of preferences is not appropriate, because welfare depends on the respective sizes of assets and liabilities. The corresponding “Asset-Liability Management” (ALM) objective reads:

\[
\max_{w} \mathbb{E} \left[ U \left( \frac{A_T}{L_T} \right) \right],
\]

s.t. short-term constraints (3.1) and budget constraint (2.7).

In this paper we do not derive the value of liabilities from a structural model of the investor’s balance sheet. We instead consider a reduced-form representation of this value as a stochastic process:

\[
\frac{dL_t}{L_t} = \left[ r_t + x_L(T-t) \right] dt + \sigma_L(T-t)^{\gamma} dz_t.
\]

Parameters \(x_L\) and \(\sigma_L\) are assumed to be deterministic functions of time-to-horizon so as to simplify the subsequent discussion on the replicability of the exchange option. These assumptions cover the case where the value of liabilities is the price of an inflation-indexed zero-coupon, as in Martellini and Milhau (2009).

We denote with \(\tilde{A}^u\) the optimal wealth process in the “unconstrained” ALM problem, i.e. in Problem (3.15) with \(F = 0\). We also denote with \(\tilde{M}^u\) the minimax pricing kernel associated to this program. The optimal unconstrained strategy is:

\[
\tilde{w}_t^u = \frac{\lambda_t^{MV}}{\gamma \sigma_t^{MV}} w_t^{MV} + \left( 1 - \frac{1}{\gamma} \right) \beta_t^L \tilde{w}_t^L
\]

\[
- \left( 1 - \frac{1}{\gamma} \right) \left[ C_L^L (T-t) + C_3 (T-t) \lambda_t^{\gamma} \right] \beta_t \tilde{w}_t^L,
\]

(3.16)

where the liability-hedging portfolio (LHP) and the beta of liabilities w.r.t. the LHP are:

\[
w_t^L = \frac{(\sigma' \sigma)^{-1} \sigma' \sigma_L (T-t)}{1'(\sigma' \sigma)^{-1} \sigma' \sigma_L (T-t)},
\]

\[
\beta_t^L = 1'(\sigma' \sigma)^{-1} \sigma' \sigma_L (T-t),
\]
3. Portfolio Choice With Performance Constraints

$C_3$ is introduced in Proposition 1 and $C_2$ is solution to an ODE written in Appendix A.5. Our guess for the optimal payoff in (3.15) is:

$$\tilde{A}_T = F_T + \left[\tilde{\xi} \tilde{A}_T^{\nu} - F_T\right]^+,$$

where the constant $\tilde{\xi}$ satisfies

$$\mathbb{E}\left[M_T^{\nu} \tilde{A}_T^{\nu}\right] = A_0.$$

As in Appendix A.2, we show that the expected utility from $\tilde{A}_T^{\nu}/L_T$ is greater than or equal the expected utility from any funding ratio $A_T/L_T$ as long as the wealth $A_T$ satisfies $A_T \geq L_T$. Hence, $\tilde{A}_T^{\nu}$ is the optimal terminal wealth in (3.15) if it is replicable. Sufficient conditions for replicability can be given as in Proposition 3: the payoff $\tilde{A}_T^{\nu}$ is attainable if the volatility of the ratio $\tilde{R}^u = \tilde{A}_T^{\nu}/F$ is progressively measurable with respect to $\mathcal{F}$ and the sources of risk $z^s$, $z^r$, $z^l$ are spanned. In particular, we do not require that liability risk be spanned. Under these assumptions, the payoff $\tilde{A}_T^{\nu}$ is replicable, the optimal wealth is

$$\tilde{A}_T^c = \tilde{\xi} \tilde{A}_T^{\nu} + \tilde{p}(t, \tilde{\xi} \tilde{R}^u_t, V_t),$$

where

$$\tilde{p}(t, \tilde{\xi} \tilde{R}^u_t, V_t) = \mathbb{E}_t^{\mathcal{F}}\left[(1 - \xi \tilde{R}^u_t)^+\right]$$

is the price of the insurance put in the numeraire F. The optimal strategy, solution to (3.15), can be written as:

$$\tilde{w}_t^c = \left(1 - \tilde{m}_t \frac{F_t}{\tilde{A}_t^c}\right) \tilde{w}_t^{\nu} + \tilde{m}_t \frac{F_t}{\tilde{A}_t^c} w_t^F + \frac{F_t}{\tilde{A}_t^c} (\sigma' \sigma)^{-1} \sigma' \sigma_y \tilde{p}_3,$$

(3.17)

where $\tilde{m}_t = \tilde{p} - \tilde{\xi} \tilde{R}^u_t \tilde{p}_2$ and $\tilde{p}_2$ and $\tilde{p}_3$ are the deltas of the put with respect to the second and third variable respectively. The differences between this ALM strategy and the Asset Management one (i.e., the one with no liabilities) are the underlying unconstrained strategy and the price of the insurance put.
4. Opportunity Cost of Insurance
In this section we study in more details the cost of insurance against downside risk and we show that it can be reduced by imposing a cap on terminal wealth.

4.1 Measuring the Cost of Insurance Against Downside Risk

The price of the insurance put is \((1 - \xi) A_0\), which shows that \(1 - \xi\) represents the relative cost of insurance against the downside risk, while \(\xi\) represents the percentage of participation to the upside. Moreover, the price at time 0 of the exchange option that pays \([\xi A^m_T - F_T]^+\) at time \(T\) is comprised between the no-arbitrage bounds:

\[\xi A_0 - F_0 < A_0 - F_0 < \xi A_0.\]  

The lower bound is the intrinsic value of the option at date 0, while the upper bound is the present value of \(\xi A^m_T\). Direct manipulation of (4.1) leads to the following result.

**Proposition 6** The relative cost of insurance, \(1 - \xi\), satisfies:

\[0 < 1 - \xi < \frac{F_0}{A_0}.\]

This proposition shows that insurance always has a positive cost, and that the cost is bounded from above by \(F_0\). The latter property makes intuitive sense: the investor could always protect the floor by purchasing a zero-coupon that delivers exactly \(F_T\) at date \(T\), and investing the remainder of his wealth in the unconstrained strategy. In this case, the insurance premium would be the price of the zero-coupon, that is \(F_0\). Equation (3.10) says that it is not optimal to do so, and Proposition 6 provides a partial explanation for this suboptimality: insurance is less expensive if one buys a put option written on the unconstrained strategy than if one invests directly in the floor.

4.2 Decreasing the Cost of Insurance Against Downside Risk

In the optimal strategy of Proposition 4, insurance against downside risk is achieved by buying a put option written on the optimal unconstrained strategy and a stochastic exercise price equal to the floor. The corresponding terminal wealth is thus:

\[A^*_T = \max(\xi A^m_T, F_T).\]

In order to decrease the cost of insurance against downside risk, one can consider giving up the part of wealth that is in excess of a cap \(C_T\) such that \(C_T \geq F_T\) almost surely. The corresponding payoff is:

\[A_T^{2c} = \xi' A^m_T + [F_T - \xi' A^m_T]^+ - [\xi' A^m_T - C_T]^+,\]

with the constant \(\xi'\) being adjusted in such a way that:

\[\mathbb{E}[M^m_T A_T^{2c}] = A_0.\]

The superscript “2c” refers to the presence of two constraints, since the payoff (4.2) satisfies:

\[F_T \leq A_T^{2c} \leq C_T\] almost surely.

In the same way as we considered replicable floors, we consider replicable caps, so that there exists a cap-replicating portfolio (CRP) such that:

\[\frac{dC_t}{C_t} = \left[ r_t + (w^C_t)' \sigma_A^t \right] dt + (w^C_t)' \sigma^C_t d\zeta_t.\]
We recall that $\mathbb{Q}^F$ defined in (3.11) is the probability measure under which asset prices in the numeraire $F$ follow martingales, and that $R^u$ is the ratio $A^u/F$. We will use similar definitions for the cap process. Let $\mathbb{Q}^C$ be the probability measure defined by

$$\frac{d\mathbb{Q}^C}{d\mathbb{P}} = \frac{M_{t, t}^u}{C_0}.$$

Under $\mathbb{Q}^C$, asset prices in the numeraire $C$ follow martingales. Let also $r^u$ denote the ratio $A^u/C$.

The following proposition shows that the payoff (4.2) is in fact optimal subject to the constraint that wealth lie between $F_t$ and $C_t$ at all dates.

**Proposition 7** Assume that:

- equity premium risk is spanned (e.g., $\rho_{\lambda} = -1$);
- the vectors $w^F_t$ and $w^C_t$ are functions of time and current values of stochastic processes $V_t$;
- the initial wealth satisfies $F_0 \leq A_0 \leq C_0$.

and consider the program:

$$\max \mathbb{E} \left[ U (A_T) \right], \text{subject to } F_t \leq A_t \leq C_t \text{ for all } t, \text{ and budget constraint (2.7), with } C_T \geq F_T \text{ almost surely.}$$

Then the optimal terminal wealth is given by (4.2), with $\xi'$ chosen so as to make (4.3) hold.

Let $p(t, \xi'R_t^u, V_t)$ and $c(t, \xi'R_t^u, V_t)$ be the prices in the numeraires $F$ and $C$ of the put and the call written on the unconstrained strategy in (4.2):

$$p(t, \xi'R_t^u, V_t) = \mathbb{E}^{Q^F}_t \left[ (1 - \xi'R_T^u) \right],$$

$$c(t, \xi'R_t^u, V_t) = \mathbb{E}^{Q^C}_t \left[ \xi'R_T^u - 1 \right].$$

Then the optimal strategy is:

$$w^{2e}_t = \left[ 1 - m'_u \frac{F_t}{A_t^{2e}} + m''_u \frac{C_t}{A_t^{2e}} \right] w^{u}_t + m'_u \frac{F_t}{A_t^{2e}} w^F_t - m''_u \frac{C_t}{A_t^{2e}} w^C_t + \frac{F_t}{A_t^{2e}} (\sigma' \sigma)^{-1} \sigma' \sigma V p_3 - \frac{C_t}{A_t^{2e}} (\sigma' \sigma)^{-1} \sigma' \sigma V c_3,$$

where:

$$m'_t = p - \xi' R_t^u p_2,$$

$$m''_t = c - \xi' R_t^u c_2.$$

**Proof.** See Appendix A.6.

In the strategy that pays off (4.2), insurance is performed by purchasing a put written on the unconstrained strategy with strike price $F_t$ and selling a call with higher strike price $C_T$. The cost of insurance is thus the price of this synthetic option strategy, that is $1 - \xi' A_0$. Appendix A.6 shows that $\xi < \xi'$, equivalently that $1 - \xi < 1 - \xi'$. Hence insurance is less expensive in strategy (4.4) than in strategy (3.13).

In practice, a common choice for $C_T$ is $\alpha F_T$ with the constant $\alpha$ taken strictly greater than 1. With this choice, the cap-replicating portfolio coincides with the floor-replicating portfolio.
4. Opportunity Cost of Insurance
5. Examples of Floors
5. Examples of Floors

In this section we review some examples of floors, with the associated constrained optimal strategy.

5.1 Zero-Coupon Floor
A typical example is when the terminal floor value is constant:

\[ F_T = k \frac{A_0}{B(0, T)}. \]

The purpose here is to protect a fraction \( k/B(0, T) \) of the initial capital \( A_0 \). Then the short-term constraints read:

\[ A_t \geq k \frac{A_0}{B(0, T)} B(t, T), \quad t \leq T. \]

The FRP is thus:

\[ w_t^F = \frac{D(T - t)}{D(tB)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

The assumptions of Proposition 4 are satisfied, provided we assume \( \rho_s = -1 \). Moreover, the volatility of \( R_t^u \) is a function of \( (t, \lambda_t^S) \) only, so the put price is a function of \( (t, R_t^u, \lambda_t^S) \).

With a slight abuse of notation, we can let

\[ \rho(t, \xi R_t^u, \lambda_t^S) = E_t^Q \left[ \left( 1 - \xi R_t^u \right)^+ \right], \]

which leads to the following re-expression for the optimal constrained strategy:

\[ w_t^{*c} = \left( 1 - m_t \frac{F_t}{A_t^{*c}} \right) w_t^{*u} + m_t \frac{F_t}{A_t^{*c}} w_t^F + p_t \theta_t \beta A_t \lambda_t^S. \]

5.2 Stock Floor
Another common choice is to choose a floor proportional to the level of the stock index:

\[ F_t = \frac{A_0}{S_0} S_t. \]

The objective pursued by imposing the short-term constraints (3.1) is to ensure that the performance of the portfolio over \([0, t]\) be at least equal to that of the stock. The FRP is entirely invested in the stock:

\[ w_t^F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Again, assumptions of Proposition 4 are satisfied if we assume \( \rho_s = -1 \). The optimal constrained strategy can then be expressed as (5.1).

5.3 Drawdown Floor
The example of drawdown (DD) constraints is highly relevant in practice, especially in private wealth management or retail money management. Indeed, investors are reluctant to experience a fall in the value of their portfolio, and want to protect a fraction of the all-time high level of their wealth. Fund managers have to take this aversion for loss into account, because a large loss in value of assets under management is likely to result in a loss of clients. The original paper by Grossman and Zhou (1993) considers a DD constraint of the form:

\[ e^{-\nu t} A_t \geq \delta \max_{0 \leq s \leq t} \left[ e^{-\nu s} A_s, M_0 \right], \quad (5.2) \]

where \( M_0 \) is a constant and \( \nu \) is a discount rate that they assume to be strictly smaller than the constant interest rate.

In Cvitanic and Karatzas (1995) and Elie and Touzi (2008), the interest rate is possibly stochastic, and the drawdown constraints read:

\[ \frac{A_t}{S_t^0} \geq \delta \max_{0 \leq s \leq t} \frac{A_s}{S_s^0}, \quad \text{for all } t, \quad (5.3) \]
which can be expressed in words as: “the current discounted wealth must not fall below a fraction \( \delta \) of its maximum-to-date”. Hence the DD floor reads:

\[
F_t = \delta S^0 \max_{0 \leq s \leq t} \frac{A_s}{S_s}.
\]

Because it is defined in terms of past values of wealth, this process is no longer exogenous, as was the generic floor (3.2). Instead, it is an endogenous floor. This has an important consequence: the floor is no longer replicable, because it is not possible to find a strategy whose discounted value is equal to the running maximum of discounted wealth. But it is still possible to find a strategy that is safe as far as the objective of respecting the constraint (5.3) is concerned: it suffices to invest in cash only, because \( A_t = S^0_t \) clearly satisfies (5.3). In other words, the drawdown floor is not replicable, but it is super-replicable.\(^4\)

### 5.4 Maximum of Two Floors

In some situations, one may want to ensure that wealth stays above two floors \( F_1 \) and \( F_2 \) at all times. In other words, one imposes the following series of short-term constraints:

\[
A_t \geq \max \left( F^1_t, F^2_t \right), \quad \text{for all } t \leq T. \tag{5.4}
\]

In the following discussion, we assume that both \( F_1 \) and \( F_2 \) are replicable. The right side in (5.4) is in general not the value of a portfolio, but under convenient assumptions, it is super-replicable. Indeed, let us assume that the payoff \( \max \left( F^1_T, F^2_T \right) \) is attainable. Since we have

\[
\max \left( F^1_T, F^2_T \right) = F^1_T + \left( F^2_T - F^1_T \right)^+,\n\]

it is equivalent to assume that the payoff \( \left( F^2_T - F^1_T \right)^+ \) is attainable. The price of this payoff is then

\[
F_t = \mathbb{E}_t \left[ \frac{M^u_t}{M^u_T} \max \left( F^1_T, F^2_T \right) \right].
\]

Since \( F_T \geq F^1_T \) and \( F_T \geq F^2_T \), we have \( F_t \geq F^1_t \) and \( F_t \geq F^2_t \) at all dates \( t \), hence:

\[
F_t \geq \max \left( F^1_t, F^2_t \right), \quad \text{for all } t \leq T,
\]

with equality at date \( T \). Hence, for (5.4) to be satisfied, it suffices to ensure that

\[
A_t \geq F_t, \quad \text{for all } t \leq T. \tag{5.5}
\]

The optimal strategy subject to this series of short-term constraints is given by Proposition 4. This discussion assumes that the payoff \( \left( F^2_T - F^1_T \right)^+ \) is attainable. As noted above, the mere replicability of \( F_1 \) and \( F_2 \) is not sufficient to ensure that the exchange option is replicable too. In order to have a replicable payoff, it suffices to assume that the volatility of the ratio \( F_2/F_1 \) is deterministic. Then, applying Margrabe’s formula, we obtain:

\[
F_t = F^1_t \mathcal{N} \left( -d^2_{1,t} \right) + F^2_t \mathcal{N} \left( d^1_{1,t} \right), \tag{5.6}
\]

with:

\[
d^1_{1,t} = \frac{1}{\sigma^F_{1,t}} \left[ \ln \frac{F^2_t}{F^1_t} + (\sigma^F_{1,t})^2 \right],
\]

\[
d^2_{1,t} = d^1_{1,t} - \sigma^F_{1,t},
\]

\[
(\sigma^F_{1,t})^2 = \int_t^T \left( \frac{\sigma^F_{2,s}/F^1_s}{F^1_t} \right)^2 ds,
\]

and \( \sigma^F_{2,F} \) is the volatility of \( F_2/F_1 \). Applying Itô’s formula to (5.6), we can write the FRP as:

\[
\omega_t^F = \frac{F^1_t}{F_t} \mathcal{N} \left( -d^F_{2,t} \right) \omega_t^{F1}
\]

\[
+ \left[ 1 - \frac{F^1_t}{F_t} \mathcal{N} \left( -d^F_{2,t} \right) \right] \omega_t^{F2}.
\]
5. Examples of Floors

The optimal strategy subject to the short-term constraints (5.5) is then given by (5.1).

5.5 Minimum of Two Caps

One may want to restrict the wealth to be lower than two caps $C_1$ and $C_2$ at all dates:

$$A_t \leq \min\left( C_t^1, C_t^2 \right) \quad \text{for all } t \leq T. \quad (5.7)$$

Even if $C_1$ and $C_2$ are the wealths generated by portfolio strategies, the right side is not the wealth generated by a portfolio strategy, so it cannot be taken as the value of the cap. However, if the payoff

$$C_T = \min\left( C_T^1, C_T^2 \right)$$

is replicable, then its price $C_t$ satisfies:

$$C_t \leq \min\left( C_t^1, C_t^2 \right) \quad \text{for all } t \leq T.$$  

Hence, the short-term constraints are satisfied if the more severe short-term constraints

$$A_t \leq C_t \quad \text{for all } t \leq T \quad (5.8)$$

are respected.

A simple sufficient condition that ensures that the payoff $C_T$ is attainable is that both $C_T^1$ and $C_T^2$ are attainable, and the volatility of the ratio $C_T^1/C_T^2$ is deterministic. Under this assumption, the value of the cap is:

$$C_t = C_t^1 \mathcal{N} \left( -d_{1,t}^C \right) + C_t^2 \mathcal{N} \left( d_{2,t}^C \right),$$

with:

$$d_{1,t}^C = \frac{1}{\sigma_{t,T}^C} \left[ \ln \left( \frac{C_t^1}{C_t^2} \right) + \frac{1}{2} \left( \sigma_{t,T}^C \right)^2 \right],$$

$$d_{2,t}^C = d_{1,t}^C - \sigma_{t,T}^C,$$

$$\left( \sigma_{t,T}^C \right)^2 = \int_t^T \left( \sigma_s^{C_1/C_2} \right)^2 \mathrm{ds}.$$
In this section, we use market data to obtain realistic parameter values, and we compare different strategies aimed at respecting short-term constraints based on their expected utilities.

### 6.1 Model Calibration

We first calibrate the model to US market data by using quarterly data from Bloomberg and CRSP. The stock index is represented by the S&P500, whose returns are available from CRSP over the period from Q4.1964 to Q4.2010. We assume that the Sharpe ratio is an affine function of the dividend yield, defined as the ratio of the sum of dividends paid over the last four quarters, over the current price index:

\[ \lambda^S_t = m \text{DY}_t + p. \]  \tag{6.1}

Data on the US yield curve is extracted from Bloomberg. We use the nominal yields on 3-month T-bills and on Government bonds of maturities 1, 3, 5 and 10 years. The observed yield of maturity \( \tau \) is denoted as by \( \tilde{y}(t, \tau) \). All yields are available over the period from Q2.1961 to Q3.2011. We assume that the observed 3M-T-bill rate is equal to the model-implied rate, so the short-term rate can be computed from the 3M-rate. In the end, our observation vector at date \( t_i \) has seven components:

\[
\begin{pmatrix}
\tilde{y}(t_i, 3M) & \tilde{y}(t_i, 1Y) & \tilde{y}(t_i, 3Y) \\
\tilde{y}(t_i, 5Y) & \tilde{y}(t_i, 10Y) & \log \left( \frac{S_{t_i}}{S_{t_i-1}} \right) & \text{DY}_t
\end{pmatrix}
\]

The state vector is the vector \( V \) defined in (2.3).

We calibrate the parameters of the model by maximising the likelihood of observations. However, we do not maximise the full likelihood of the observations \( \tilde{Z}_{t_1}, \ldots, \tilde{Z}_{t_n} \). Instead, we first calibrate the interest rate parameters \( a, b, \sigma, \) and \( \lambda \), by maximising the likelihood of the yields, and then we calibrate the stock and Sharpe ratio parameters \( \sigma^S, \kappa, \lambda, \sigma^r, \rho_{Sr}, \rho_{Sa} \) and \( \rho_{r\lambda} \) by maximising the likelihood of stock returns and dividend yields. This sequential procedure enables to reduce the number of parameters to estimate in each numerical maximisation, which helps reduce numerical concerns. Moreover, it avoids creating cross-dependencies between parameter estimates. For instance, the estimates for the interest rate parameters are not impacted by the series of returns chosen for the stock index, nor they are affected by the way the dividend yield is computed. Details on the procedure can be found in Martellini and Milhau (2011).

Parameter estimates are reported in Table 1. We have also estimated the standard deviations of Maximum Likelihood (ML) estimators by taking the square roots of the diagonal elements of the inverse of the Hessian matrix, taken as a consistent estimator of Fisher’s information matrix. As is typical in such calibrations, parameters that relate to expected returns are much more imprecisely estimated than volatility parameters. For instance, the standard deviation of the ML estimator for \( b \) is larger than the estimate itself. The Sharpe ratio of nominal bonds, estimated to be 33.85%, comes also with a high standard deviation of 15.92%, so the asymptotic confidence interval at the 95% level is as wide as [2.64%, 65.06%]. The long-term level of the Sharpe ratio of the stock, estimated at 44.36%, looks to be estimated with a satisfactory precision since the standard deviation is only 5.70%. But the true value of the current Sharpe ratio, \( \lambda^S_t \), is...

hard to compute in practice, because the parameters $m$ and $p$ that relate it to the dividend yield (see (6.1)) are estimated with poor precision: the standard deviations of these estimators are about one half of the estimates. Finally, the correlation $\rho_{SL}$ is found to be strongly negative ($-94.94\%$), which provides empirical justification for the assumption that $\rho_{SL} = -1$, which we will assume to hold in the remainder of this section.

6.2 Framework of Numerical Exercise

We choose the zero-coupon floor (see Section 5), so the short-term constraints are:

$$A_t \geq F_t \equiv k \frac{A_0}{B(0, T)} B(t, T), \text{ for all } t \leq T.$$ 

With this choice, we have $F_0 = kA_0$, and the FRP is given by:

$$\omega_i^F = \frac{D(T-t)}{D(T_B)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In order to compute expected utilities, we use Monte Carlo simulations. By considering $\rho_{SL} = -1$, the market is complete and therefore there exists a unique risk-neutral probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$. We implement the optimal unconstrained strategy of Proposition 1 on a monthly basis, and simulate 30,000 trajectories under the risk-neutral measure $\mathbb{Q}$, so as to obtain outcomes for the optimal unconstrained wealth $A_T^{nu}$. Then, we can compute $\xi$ by numerically solving the equation

$$\mathbb{E}^\mathbb{Q} \left[ e^{-\int_0^T r_s ds} (\xi A_T^{nu} - F_T)^+ \right] = A_0 - F_0,$$

where the expected discounted payoff is approximated with our Monte Carlo simulations. Having computed $\xi$, we can easily derive the payoff of the optimal constrained strategy of Proposition 4 as $A_T^{nc} = \max [\xi, A_T^{nu}, F_T]$. Finally, we can compute the expected CRRA utilities from $A_T^{nu}$ and $A_T^{nc}$ by simulating the change of measure $Z_T$ as

$$Z_T = \frac{d\mathbb{P}}{d\mathbb{Q}} = \exp \left[ -\int_0^T \frac{\lambda_s}{2} ds + \int_0^T \lambda_s dz_s \right],$$

and using the following identity $\mathbb{E}[U(A_T)] = \mathbb{E}^\mathbb{Q}[Z_T U(A_T)]$.

When we refer to the three investors: aggressive, moderate and defensive, we refer to the calibration of three risk aversion values reflecting different risk behaviours. The aggressive investor holds on average, over a 20-year investment horizon, 75% of equity in his portfolio, the moderate investor holds 50% of equity and the defensive 25%. These equity positions lead to the following risk aversion values: $\gamma = 19, 32$ and 70.

6.3 Opportunity Cost of Imposing Short-Term Constraints

Our first objective is to measure the cost of optimally taking into account the short-term constraints. To do so, we compute the value of the insurance put $1 - \xi$ as a function of the risk aversion $\gamma$ and of the initial ratio $k = \frac{F_0}{A_0}$. Figure 1 shows that the cost of insurance is a decreasing function of risk aversion. This result was expected, since the investor who follows a risky strategy will pay a higher price for his insurance against floor violations than the investor who follows a non-risky strategy. The difference between the two graphs of Figure 1 also illustrates that when the investment horizon increases, insurance becomes more costly. On Figure 2, we can see that the cost of insurance

is an increasing function of $k$. This can be explained by the fact that when $k$ gets closer to 1, the dollar amount allocated to the unconstrained strategy shrinks to zero, so the access to the upside of this strategy is lost.

Our second objective is to measure the utility loss induced by the constrained strategy with respect to the unconstrained one. We thus compute the monetary utility loss (MUL) of the optimal constrained strategy (3.13) with respect to the optimal unconstrained one, (2.10). Formally, the MUL $x$ is defined as the amount by which the initial investment in the unconstrained strategy must be decreased for the indirect utility to be equal to the expected utility from the constrained one. The investor is thus indifferent between: (a) invest $A_0$ in the constrained strategy; (b) invest $A_0 - x$ in the unconstrained one. If we denote with $EU^c(A_0)$ the expected utility achieved with the constrained strategy and an initial investment $A_0$, and with $EU^u(A_0)$ the expected utility achieved with the unconstrained strategy and the initial capital $A_0$, we have, by definition:

$$EU^u(A_0 - x) = EU^c(A_0).$$

Using (2.12), we obtain:

$$x = A_0 \left[ \left( \frac{EU^c(A_0)}{EU^u(A_0)} \right) \frac{1}{1 - \gamma} - 1 \right].$$

As for the cost of insurance, we want to measure the utility loss of short-term constraints for different values of the risk aversion $\gamma$ and initial ratios $k$.

The curves in Figures 3 and 4 are very similar to those in Figures 1 and 2. This makes sense: when the cost of insurance is high, the coefficient $\xi$ is low, so the constrained wealth is very different from the unconstrained one, and the MUL is high. On the other hand, if the cost of insurance is close to zero, the constrained and the unconstrained strategies are very similar, which results in a low MUL. More surprisingly, the values of the MUL appear to be very close to the costs of insurance $1 - \xi$, which shows that either measure can be taken as a valuable proxy of the other.

We also measure the opportunity cost of short-term constraints for different values of $\sigma_\lambda$, the volatility of the Sharpe ratio. The purpose is to see whether short-term constraints are less costly in the presence of a low variability of the equity Sharpe ratio. Figure 5 shows indeed that this is the case. Considering smaller values of $\sigma_\lambda$, implies that the hedging demand with respect to the Sharpe ratio decreases, and so does the equity weight in the unconstrained strategy. This leads to an unconstrained strategy mostly invested in the zero-coupon, which happens to be the floor that we have considered in our numerical experiments. Therefore, the constrained and unconstrained strategies get closer to the floor when $\sigma_\lambda$ goes to 0, leading to similar expected utilities.

We then study the impact of imposing a cap on the wealth in addition to the floor. This additional constraint, as shown in Section 4.2, is supposed to decrease the cost of insurance, $1 - \xi'$. The top panel of Figure 6 illustrates this decrease of the cost as a function of the initial value of the cap constraint $C_0 = k'A_0$. We notice that the cost can become negative for values of $k'$ that are close to 1. Indeed, in that case, the insurer caps an important part of the profits

because he delivers to the investor at most $C_T$, enabling him to pay for the floor insurance $1 - \xi$, and make additional profits. In practice, such cases will be rare if $k'$ is significantly higher than $k$. We can also notice that as $k'$ increases, the cost $1 - \xi'$ converges to the cost of the floor insurance $1 - \xi$ that we reported for $k = 90\%$ in Figure 2. Finally, we can conclude that the reduction of the insurance cost comes at a certain price in terms of expected utility. Indeed, cutting off the right tail of the optimal constrained distribution increases the MUL as illustrated in the bottom panel of Figure 6.

6.4 Cost of Inappropriate Management of Short-Term Constraints

Managing short-term constraints can be done in various ways. Therefore, we propose to compare the welfare achieved by the optimal constrained strategy (3.13), to the one attained with strategies that manage short-term constraints in a sub-optimal way. We previously observed that this can be done by comparing monetary utility losses of any constrained strategies (optimal and sub-optimal).

6.4.1 Wrong Building Blocks and Wrong Allocation

In this section, we consider strategies that use neither the good building blocks (floor-replicating portfolio versus stocks instead of floor-replicating portfolio versus unconstrained strategy), nor the good dynamic allocation (CPPI and fixed-mix instead of the constrained OBPI strategy). More precisely, we will consider the following two strategies:

- A fixed-mix (FM) allocated to the stock and the floor-replicating portfolio:
  \[ w_{t}^{fm} = x(k)w_t^S + (1 - x(k))w_t^F, \quad (6.2) \]
  where, for each $k$, $x(k)$ is calibrated to obtain the highest exposure to equities while maintaining a probability of breaching the floor over the investment period less than 0.5%. Note that for $x = 0$, the portfolio coincides with the FRP, hence, if the initial capital $A_0$ satisfies $A_0 \geq F_0$, the constraint $A_t \geq F_t$ will be satisfied with probability 1.

- A constant proportion portfolio insurance (CPPI) allocated to the stock and the floor-replicating portfolio:
  \[ w_{t}^{cppi} = m \left( 1 - \frac{F_t}{A_t^{cppi}} \right) w_t^S + \left[ 1 - m \left( 1 - \frac{F_t}{A_t^{cppi}} \right) \right] w_t^F, \quad (6.3) \]
  where $m$ is a multiplicative factor introduced to leverage the stock position.

The wealths generated from the above strategies are respectively given in Appendix A.7 and A.8, where it is easy to check that, by construction, the CPPI strategy generates a wealth that always remains above the floor. Note that the particular case $m = 1$ leads to the simple wealth, $A_t^{cppi} = F_t + (A_0 - F_0) \frac{S_t}{S_0}$, which reduces to a buy-and-hold strategy where the investor holds one unit of the floor (the zero-coupon bond here), and invests his remaining wealth $(A_0 - F_0)$ in the stock.

In Figures 7 and 8, we compute the monetary utility losses (MUL) of the above FM and CPPI strategies and compare them to the MUL of the optimal constrained strategies. All MUL values are computed with respect to the optimal unconstrained
strategies. In each graph, the MULs of the non-optimal strategies are represented by the black dotted line.

One can conclude that using the strategies based on the wrong building blocks and the wrong dynamic asset allocation (strategies that are often proposed by the industry) leads to significant utility losses. Indeed, for low values of \( k \), the losses can reach 100% of the wealth (for both the FM and CPPI strategies). When \( k \) increases, utility losses decrease but always remain above the blue line which represents the losses of the optimal constrained strategies.

These results hold for any type of investor, whether aggressive, moderate or defensive. Indeed, for the defensive investor, MULs are slightly lower, but the gain of using the optimal constrained strategy instead of the sub-optimal strategies remains high.

In Figure 7, we also notice that the utility losses increase with the multiplicative factor \( m \) of the CPPI strategy. The position in stocks increases with \( m \), thus for high values of \( m \), these strategies are highly invested in stocks instead of being invested in the optimal unconstrained strategy, leading to higher loss of utility. On the other hand, for small values of \( m \), the CPPI is invested in the FRP, which has higher expected utility than the stock.

6.4.2 Right Building Blocks and Wrong Allocation

In the following, we consider strategies that use the good building blocks (floor-replicating portfolio versus unconstrained strategy), but the wrong dynamic allocation between both building blocks (CPPI and fixed-mix instead of the constrained OBPI strategy). More precisely, we will consider the following two strategies:

- An extended fixed-mix (ExFM) allocated to the unconstrained optimal strategy and the floor-replicating portfolio:
  \[
  w_{t}^{extfm} = x(k)w_{t}^{*u} + (1 - x(k))w_{t}^{F},
  \]
  where, for each \( k \), \( x(k) \) is calibrated to obtain the highest exposure to the unconstrained strategy while maintaining a probability of breaching the floor over the investment period less than 0.5%.

- An extended constant proportion portfolio insurance (ExCPPI) allocated to the unconstrained optimal strategy and the floor-replicating portfolio:
  \[
  w_{t}^{exeppi} = m \left( 1 - \frac{F_{t}}{A_{t}^{exeppi}} \right) w_{t}^{*u} + \left[ 1 - m \left( 1 - \frac{F_{t}}{A_{t}^{exeppi}} \right) \right] w_{t}^{F}.
  \]

where \( m \) is a multiplicative factor introduced to leverage the position in the unconstrained strategy.

In Figures 7 and 8, we compute the monetary utility losses (MUL) of the above Extended FM and CPPI strategies and compare them to the MUL of the optimal constrained strategies. All MULs are computed with respect to the optimal unconstrained strategies. In each graph, the MULs of the non-optimal strategies are represented by the green dotted line.

As expected, by using the right building blocks, the MUL of the extended sub-optimal strategies have decreased significantly.

compared to the MUL of the classical FM and CPPI strategies presented in the previous section. Nonetheless, as for the classical FM and CPPI strategies, the MULs of these extended sub-optimal strategies remain higher than those of the optimal constrained strategy, except for low values of $k$ where the optimal unconstrained wealth is always (in our 30,000 Monte Carlo simulations) higher than the floor, leading to trivial calibrated values $x(k) = 0$. These exceptions are therefore artificial, and have no real financial interest since there is obviously no need for insurance when $k$ is very low.

In contrast with the classical CPPI, the MUL of the extended strategy in Figure 7 does not always increase with the multiplicative factor $m$ of the strategy (particularly for high values of $k$). Another interesting remark is that the MUL of the extended CPPI strategy for $m = 1$ has a very different pattern than the ones computed with higher values of $m$. It is indeed similar to the MUL of the extended FM and the optimal constrained strategies since it increases with $k$. More specifically, the MUL starts at 0 (for $k = 0$), and increases to the MUL of the FRP for $k = 1$. These observations remain true for any type of investor: aggressive, moderate or defensive.
7. Numerical Analysis of the Benefits of Optimal Risk-Controlled Strategies with Maximum Drawdown Floor
7. Numerical Analysis of the Benefits of Optimal Risk-Controlled Strategies with Maximum Drawdown Floor

In this section, we present another numerical illustration of the benefits of risk-controlled strategies. This illustration is meant to provide a closest approximation to a real-world implementation of the dynamic asset allocation strategy, and as such it differs from the previous one on a number of fronts. First, the floor we use is not a capital guarantee floor, but instead a maximum drawdown floor as introduced in Subsection 5.3. For this illustration, we also consider for the sake of increased realism a more general model with stochastic equity volatility. As recalled in Section 2, the presence of stochastic volatility does not justify the introduction of a separate hedging demand, so it can be accommodated in the model without a significant increase in complexity. Following Heston (1993), we assume that the conditional variance of stock returns follows a square-root process:

\[ dV_t = \alpha (\bar{V} - V_t) \, dt + \sigma_V \sqrt{V_t} \, dz^V_t, \]

where \( \bar{V} \) represents the long-term volatility, \( \sigma_V \) the volatility of volatility, and \( \alpha \) the speed of reversion of the equity volatility to its long-term value. Finally, we borrow base case parameter values, reported in Table 2, from see Amenc et al. (2012), who calibrate a similar model to European data.

7.1 Opportunity Cost from Imposing Max Drawdown Constraints

In what follows, we provide an illustration of the benefits of risk-controlled strategies. We first implement unconstrained strategies, taking the time-horizon to be equal to 20 years, while the risk aversion parameter \( \gamma \), which is not observable, is calibrated in such a way that the average allocation to equity over the 20-year life of the strategy is equal to a target of 10%, 20% or 30%. The three corresponding long-term unconstrained strategies will be referred to as defensive (\( \gamma = 55 \) leading to an average stock weight of 9.95%), moderate (\( \gamma = 23 \) leading to an average stock weight of 19.90%) and aggressive (\( \gamma = 12 \) leading to an average stock weight of 30.76%) respectively. It should be noted that the allocation to the stock at a given date can be substantially higher than the average allocation, especially at the beginning of the period. The top panel of Figure 9 shows the resulting distribution of unconstrained terminal wealth for various risk aversion levels. We find the usual risk-return trade-off: strategies implemented by less risk-averse investors will contain a higher allocation to equities, which will result in a higher average wealth level as well as a higher uncertainty around the terminal wealth level.

Next, we provide an objective measure of the opportunity cost associated with a higher risk-aversion level as the additional initial contribution \( x \) needed to reach with a less aggressive strategy the same average wealth level as with a more aggressive strategy. This definition is similar to the definition of MUL, except that it concerns the expected terminal wealth instead of the expected utility of terminal wealth, and as such provides an easily interpretable measure for opportunity costs/gains. Table 4 shows that a defensive investor who wants to have the same expected wealth level as an aggressive investor needs to invest 32.51% more at initial time 0.
7. Numerical Analysis of the Benefits of Optimal Risk-Controlled Strategies with Maximum Drawdown Floor

While long-term strategies are engineered to achieve optimal risk/return trade-off over the long term, short-term losses and drawdown levels can remain substantial, especially for the aggressive investor (see Table 3). On the other hand, choosing a high risk aversion level instead of a low risk aversion level induces, as reported above, a cost of 32.51%, which can seem prohibitive. A less costly solution is to use insurance, as opposed to hedging, to manage downside risk. To do this, we implement a strategy designed to control downside risk, where the maximum drawdown is equal to $1 - \delta$ and can be set independently from the risk aversion $\gamma$. In Table 5, we consider maximum drawdown levels at 20%, 15% and 10%, respectively. We observe that the aggressive strategy implemented with a constraint of maximum drawdown at 15% is able to maintain the maximum drawdown at a value of 14.52%. This may seem a limited reduction in downside risk in view of the 15.16% maximum drawdown achieved with the unconstrained defensive strategy, but we also observe that the low target wealth of the constrained aggressive strategy is higher than the unconstrained defensive one, whatever the value of the maximum drawdown. This result makes a strong case for the management of short-term constraints through dynamic risk budgeting rather than through the choice of unnecessarily conservative investment policies.

We finally provide an objective assessment of the opportunity cost of imposing stricter drawdown constraints. Table 6 shows the additional initial contribution that one needs to invest in a given long-term strategy in order to reach the same expected wealth level as with a strategy with a looser drawdown constraint, or even with no such constraint at all. We already noticed that choosing a defensive benchmark so as to respect a 15% max drawdown constraint is not the appropriate way to manage short-term losses, since it involves a strong opportunity cost (32.51% of initial wealth for an aggressive investor). Instead, proposing an aggressive benchmark with a maximum drawdown constraint of 15% only involves an opportunity cost of 5.38% with respect to the situation where no constraint is imposed. This value is by far smaller than 32.51%, so the opportunity cost of imposing a maximum drawdown constraint is not prohibitive.

These results illustrate that not disentangling risk- and loss-aversion can have profound negative consequences for the investor, and may lead to poor investment decisions. Simple solutions exist and can be implemented directly into the strategy in order to control short-term risks while maintaining long-term performances. The bottom panel of Figure 9 shows the resulting distribution of terminal wealth for various risk and loss aversions. It is interesting to note that the left tail of the distribution of strategies implemented with short-term constraints does not vanish even though the drawdown significantly decreases, which illustrates that limiting the size of short-term losses is not equivalent to the imposition of a minimum terminal wealth.
7. Numerical Analysis of the Benefits of Optimal Risk-Controlled Strategies with Maximum Drawdown Floor

7.2 On the Tension between Long-Horizons and Short-Term Constraints

We know that the correlation between realised equity returns and expected equity returns is strongly negative, which supports the assumption $\rho_{SA} = -1$, a simplifying assumption ensuring market completeness. In this context, the unconstrained strategy has a counter-cyclical component, namely, when the stock index value decreases, then the Sharpe ratio increases, leading to an increase in the equity holding. On the other hand, taking into account short-term constraints introduces a risk budget that increases during bull markets and decreases during bear markets. This pro-cyclical effect conflicts with the aforementioned counter-cyclical effect.

For estimating the current value of the Sharpe ratio, we proceed as follows. We first define the return between the moving average stock index value over the last 3 years, denoted by $\bar{S}_t$, and the current value $S_t$ as:

$$R_t = \frac{S_t - \bar{S}_t}{S_t}$$

Then, we estimate the stock index Sharpe ratio according to the following rule: if $R_t > 10\%$, we set $\lambda^S_t = \lambda^{low}$, if $R_t < 10\%$, we set $\lambda^S_t = \lambda^{high}$, otherwise we set $\lambda^S_t = \lambda^{mid}$. Obviously, other approaches can be used as well. In particular, a standard approach in the academic literature (see for example Campbell and Viceira (1999)) consists in assuming that the Sharpe ratio is an affine function of the dividend yield (or price earning ratio).

In Figure 10, we represent the asset allocations for a 15-year investment horizon with and without maximum drawdown constraints. Panel (a) illustrates the counter-cyclical effect in the absence of the short-term constraints. Indeed, we see that the equity allocation increases during the Dot-com bubble and the subprime crisis, due to the recognition that expected return increases after a sharp decrease in realised equity returns. However, when drawdown constraints are introduced, the allocation to equity is also impacted by the diminishing risk budget. Indeed, we see that around 2000, during the Dot-com bubble, the investment in equity is lower in panel (b) than it is in panel (a). The resulting net decrease in equity allocation is compensated by a corresponding increase in cash, which is the safe super-replicating portfolio with respect to the drawdown floor. The same observation holds during the subprime crisis, where we see that the weight in equity decreases close to 20% for the strategy with maximum drawdown constraint, and then increases back to 25% toward the end of the crisis, with a net impact shifting from a pro-cyclical effect to a counter-cyclical effect.

7.3 Implementation Challenges

In this section, we discuss some practical issues related to the computation of optimal weights and implementation of the dynamic asset allocation strategy. As explained in Section 3.5, the procedure of constant proportion portfolio insurance techniques, originally designed to ensure the respect of absolute performance, can be extended to a relative return context. The techniques of traditional CPPI still apply, provided that the risky asset is re-interpreted as the unconstrained strategy, and the risk-free asset is re-interpreted as the floor-replicating portfolio, which contains...
no relative risk with respect to the floor itself. In general, investors endowed with consumption/liability objectives need to invest in two distinct portfolios: the performance-seeking portfolio (PSP) and the liability-hedging portfolio (LHP), with an allocation to PSP that is increasing in the PSP Sharpe ratio and decreasing in the investor’s risk-aversion and PSP volatility. The novelty is that the allocation to PSP versus LHP is also a function of the risk-budget, expressed in terms of distance between the asset value \( A_t \) and a (probability-weighted) floor \( F_t \), typically known as the cushion in CPPI terminology. When the margin for error disappears, i.e., when the investor’s short-term risk budget is spent, then the allocation to the PSP becomes zero, as it should. As shown in (3.17), if the volatility of the optimal unconstrained funding ratio is deterministic, then the probability weights applying to the floor are available in analytical form. In general, assuming more complex dynamics for the key state variables, no analytical expression is available for the optimal strategy (3.17). To obtain the optimal portfolio strategy, one needs to adopt a numerical procedure. One way to proceed is the following:

1. Compute the coefficient \( \xi \), which measures the access to the performance of the unconstrained strategy, or equivalently \( 1 - \xi \), which measures the cost of insurance;
2. At each rebalancing date \( t \), compute the value of the unconstrained strategy that is implied by observed quantities (value of the constrained portfolio, of the floor, and of the state variables that impact the floor-replicating portfolio or FRP);
3. Using the previously computed value \( \tilde{A}_t^u \), compute the deltas of the insurance put with respect to the floor and the state variables;
4. Compute the weights of the optimal constrained strategy through (3.17).

We now explain in more detail how to proceed for each step. For Step 1, we use the procedure described in Section 6. First, we define the terminal payoff

\[
A_T^c = F_T + [\xi A_T^u - F_T]^+,
\]

and we look for the value of \( \xi \) that makes the budget constraint hold:

\[
\mathbb{E}^Q \left[ e^{-\int_0^T r_s dt} A_T^c \right] = A_0. \quad (7.2)
\]

In other words, we find the value \( \xi \) so as to satisfy the budget constraint stating that the terminal payoff needs to be achieved with a given initial investment. In general, no analytical expression is available for the expectation on the left-hand side. Hence, one needs to use Monte-Carlo simulations, to generate outcomes for the value of the unconstrained strategy (with value \( A_0 \) at date 0), for the floor and for the short-term rate. These simulations have to be run under the equivalent martingale measure \( Q \). Having simulated \( A_T^c \) and the discount factor, one can evaluate the left-hand side of (7.2) for a given \( \xi \). It then suffices to numerically solve the value of \( \xi \) that makes (7.2) hold (this can be done for instance by dichotomy: starting from an initial value for \( \xi \), one tries a lower value if the left-hand side is greater than the right-hand side, or a greater value otherwise).

At Step 2, we are at a rebalancing date \( t \), where we observe the value of the constrained portfolio \( \tilde{A}_t^c \), of the floor \( F_t \) and of the state variables \( V_t \). We must compute the value \( \tilde{A}_t^c \) of the unconstrained strategy. It is implicitly given by the equality
7. Numerical Analysis of the Benefits of Optimal Risk-Controlled Strategies with Maximum Drawdown Floor

\[ \tilde{A}_T^u = \xi \tilde{A}_T^u + \mathbb{E}_T^Q \left[ e^{-\int_T^T r_s \, ds} \left( \xi \tilde{A}_T^u - F_T \right)^+ \right]. \]

Note that the value of \( \xi \) has already been obtained at Step 1. Thus, to estimate the price of the put in the right-hand side, it suffices to generate scenarios for \( \tilde{A}_T^u, F_T \) and the short-term rate under \( Q \). The starting value of the unconstrained strategy, \( \tilde{A}_T^u \), is precisely the quantity that we seek to obtain, and it can again be found by dichotomy. Note that \( \tilde{A}_T^u \) is homogeneous in the initial wealth \( A_T^u \), as a consequence of the weights being independent from wealth (see (3.16)). Hence it suffices to run one set of Monte-Carlo simulations with an initial value of $1 for the unconstrained strategy, and then see the price of the put as a function of \( \tilde{A}_T^u \).

Step 3 is a standard problem of estimation of Greeks. We must estimate the deltas of the put option (denoted as \( p_2 \) and \( p_3 \) in (3.17)) conditional on the observed values for the value of the floor \( F_t \) and of the state variables \( V_t \), and on the value of the unconstrained strategy obtained at Step 2. The two usual approaches are finite differences (where one uses again Monte-Carlo simulations to compute the price of the put for two "close" values of a variable), and partial differential equations. The former approach should be preferred if the price of the put is a function of more than three variables. For the latter approach, one can use the technique developed by Heston (1993) to price European options written on an underlying with stochastic volatility.
8. Conclusion
The two motives behind dynamic asset allocation decisions, namely the insurance and the hedging motives, are often perceived as inconsistent and mutually exclusive. In particular, it is often argued that dynamic insurance strategies, which typically imply a reduction to equity allocation after a market downturn, are intrinsically pro-cyclical and miss the opportunity to invest in equities when they are particularly inexpensive. We cast new light on the debate by showing that short-term constraints and long-term investing need not be mutually exclusive and can naturally coexist within the context of a long-term investing strategy that respects short-term performance constraints. Our empirical analysis has two important implications. First, it shows that in order to respect short-term constraints, a strategy with no risk budgeting, such as a fixed-mix strategy or even a utility-maximising strategy ignoring the presence of these constraints, must be very conservative: to ensure that wealth will stay above the floor, the weight allocated to performance assets must be reduced, which has a negative effect on performance. Instead, optimal risk-controlled strategies respect the constraints while opening wider access to the performance block. Second, we show that optimal strategies yield higher expected utility than sub-optimal risk-controlled strategies such as CPPI. However, they may be more difficult to implement in practice, because they involve the pricing of an insurance put and the computation of its deltas.

While our analysis covers a wide range of floors (exogenous floors, drawdown floors, etc.) and objectives (focus on nominal wealth or funding ratio), it would be desirable to investigate extensions of our results where the floor is not replicable by the traded assets. In this case, the existence of a solution to the constrained problem is not trivial, and the literature on superreplication of contingent claims might help on this topic. Another worthwhile extension would be to assess how the presence of income risk would impact our analysis of the tension between long-horizon objectives and short-term constraints.

8. Conclusion
Appendix

A. Proof of the Main Propositions

A.1 Proof of Proposition 1

Following He and Pearson (1991), we write one static optimisation program for each pricing kernel $M$

$$\max_X \mathbb{E} [U(X)], \text{ s.t. } \mathbb{E} [M_T X] = A_0,$$

where $M$ denotes a generic pricing kernel (see (2.6)). The first-order optimality condition in the static problem implies that:

$$X^* = -\frac{A_0}{\mathbb{E} \left[ M_T^{-\frac{1}{\gamma}} \right]} M_T^{-\frac{1}{\gamma}}. \quad (A.1)$$

$X^*$ is a candidate optimal payoff, but it may not be replicable. The minimax pricing kernel $M^*u$ is obtained by letting $\nu = \nu^*u$ in (2.6), where $\nu^*u$ is such that $X^*$ is replicable. The optimal wealth process is given by:

$$A^*_{t,u} = \eta_0 \mathbb{E}_t \left[ \frac{(M^*_{T^u})^{\frac{1-1}{\gamma}}}{M^*_{t^u}} \right], \quad t \leq T,$$

with:

$$\eta_0 = \frac{A_0}{\mathbb{E} \left[ (M^*_{T^u})^{\frac{1-1}{\gamma}} \right]}.$$

The optimal wealth can be rewritten as:

$$A^*_{t,u} = \eta_0 (M^*_{t^u})^{-\frac{1}{\gamma}} B(t, T)^{\frac{1-1}{\gamma}} \mathbb{E}_t \left[ \frac{(M^*_{T^u} B(T, T))^{\frac{1-1}{\gamma}}}{M^*_{t^u} B(t, T)} \right]. \quad (A.2)$$

We denote the conditional expectation in the right side with $G_t$ and search for an exponential affine representation of $G_t$:

$$G_t = \exp \left[ \frac{1 - \gamma}{\gamma} \left[ C_1 (T - t) + C_2 (T - t) \lambda_t^S + \frac{1}{2} C_3 (T - t) \left( \lambda_t^S \right)^2 \right] \right].$$

Applying Ito’s lemma to (A.2), we obtain the volatility vector of $A^*_{t,u}$:

$$\sigma_{t,u}^A = \frac{1}{\gamma} (\lambda_t + \nu_t^*u) + \left( 1 - \frac{1}{\gamma} \right) \sigma_B (T - t) - \left( 1 - \frac{1}{\gamma} \right) \left[ C_2 (T - t) + C_3 (T - t) \lambda_t^S \right] \sigma_\lambda. \quad (A.3)$$

The optimal portfolio at date $t$ is thus:

$$w^*_{t,u} = (\sigma' \sigma)^{-1} \sigma' \omega^*_{t,u}.$$
Appendix

It remains to compute the functions $C_1$, $C_2$ and $C_3$. To do this, we use the martingale property of the process $M_t^u A_t^u$: the drift of this process is zero at all dates and in almost all states of the world. Ito’s lemma shows that the condition of a zero drift is equivalent to:

\[
0 = \frac{1}{2\gamma} \left(1 - \frac{1}{\gamma}\right) \|\sigma_B - \lambda_t - \nu_t^u\|^2 + \frac{1 - \gamma}{\gamma} (C_2 + C_3\lambda_t^S) \kappa (\lambda_t - \lambda_t^S) \\
+ \frac{1}{2} \left[ \left(1 - \frac{1}{\gamma}\right) \left(C_2 + C_3\lambda_t^S\right)^2 + \frac{1 - \gamma}{\gamma} (C_2 + C_3\lambda_t^S) C_3 \right] \sigma_t^2 \\
+ \left(1 - \frac{1}{\gamma}\right) \frac{1 - \gamma}{\gamma} (C_2 + C_3\lambda_t^S) \lambda_t \left(\sigma_B - \lambda_t - \nu_t^u\right) + \left(1 - \frac{1}{\gamma}\right) \left[ \dot{C}_1 + \dot{C}_2\lambda_t^S + \frac{1}{2} \dot{C}_3 (\lambda_t^S)^2 \right].
\]

\[(A.4)\]

(We have omitted the argument $T - t$ of the functions $C_2$, $C_3$ and $\sigma_B$ in order to alleviate notations). Moreover, the volatility vector $\sigma_t^A u$ is spanned by the volatility matrix, so that $\mathbf{N} \sigma_t^A u = 0$. Using (A.3), we obtain that:

\[\nu_t^u = -(1 - \gamma) \left[C_3 + C_3\lambda_t^S\right] \mathbf{N} \sigma_t^A.\]

Plugging this expression back into (A.4), we obtain:

\[
0 = \frac{1}{2\gamma} \left(1 - \frac{1}{\gamma}\right) \|\sigma_B - \lambda_t^S \lambda_2 + (1 - \gamma) \left[C_3 + C_3\lambda_t^S\right] \mathbf{N} \sigma_t^A\|^2 \\
+ \frac{1 - \gamma}{\gamma} (C_2 + C_3\lambda_t^S) \kappa (\lambda_t - \lambda_t^S) + \left(1 - \frac{1}{\gamma}\right) \left[ \dot{C}_1 + \dot{C}_2\lambda_t^S + \frac{1}{2} \dot{C}_3 (\lambda_t^S)^2 \right] \\
+ \frac{1}{2} \left[ \left(1 - \frac{1}{\gamma}\right) \left(C_2 + C_3\lambda_t^S\right)^2 + \frac{1 - \gamma}{\gamma} (C_2 + C_3\lambda_t^S) C_3 \right] \sigma_t^2 \\
+ \left(1 - \frac{1}{\gamma}\right) \frac{1 - \gamma}{\gamma} (C_2 + C_3\lambda_t^S) \lambda_t \left[\sigma_B - \lambda_t^S \lambda_2 + (1 - \gamma) \left(C_3 + C_3\lambda_t^S\right) \mathbf{N} \sigma_t^A \right].
\]

The right side is a quadratic function of $\lambda_t^S$, with three time-dependent coefficients: one intercept, one coefficient of $\lambda_t^S$ and one coefficient of $(\lambda_t^S)^2$. Since this function must be zero almost surely at all dates, the coefficients must be zero at all dates too. These three conditions lead to the following system of ODEs:

\[
\dot{C}_3(s) = \frac{\|\lambda_3\|^2}{\gamma} + 2 \left[\frac{1 - \gamma}{\gamma} \lambda_2 C_3(s) - \kappa \right] C_3(s) + \frac{1 - \gamma}{\gamma} \left[\sigma_t^2 - (1 - \gamma) \sigma_t^4 \mathbf{N} \sigma_t^A \right] C_3(s)^2,
\]

\[
\dot{C}_2(s) = \frac{\lambda_2}{\gamma} - \frac{1 - \gamma}{\gamma} \lambda_2 \sigma_B(s) + \left[\frac{1 - \gamma}{\gamma} \lambda_2 C_2(s) - \kappa \right] C_2(s) \\
+ \left[\kappa \lambda_t^S + \frac{1 - \gamma}{\gamma} \sigma_t^4 \left[\lambda_1 - \sigma_B(s) \right] \right] C_3(s) + \frac{1 - \gamma}{\gamma} \left[\sigma_t^2 - (1 - \gamma) \sigma_t^4 \mathbf{N} \sigma_t^A \right] C_2(s) C_3(s),
\]

\[
\dot{C}_1(s) = \frac{\|\lambda_1(s) - \sigma_B(s)\|^2}{2\gamma} + \frac{1}{2} \sigma_t^2 C_3(s) + \left[\kappa \lambda_t^S + \frac{1 - \gamma}{\gamma} \sigma_t^4 \lambda_2 C_3(s) - \sigma_B(s) \right] C_2(s) \\
+ \frac{1 - \gamma}{2\gamma} \left[\sigma_t^2 - (1 - \gamma) \sigma_t^4 \mathbf{N} \sigma_t^A \right] C_2(s)^2.
\]
with the initial conditions $C_1(0) = C_2(0) = C_3(0) = 0$.

The indirect utility at date $t$ is given by:

$$
\mathbb{E}_t [U (A_T^{su})] = \frac{1}{1 - \gamma} \eta_0^{1 - \gamma} \mathbb{E}_t \left[ (M_T^{su})^{1 - \frac{1}{\gamma}} \right] 
= \frac{1}{1 - \gamma} \eta_0^{1 - \gamma} G_t B(t, T)^{1 - \frac{1}{\gamma}} (M_t^{su})^{1 - \frac{1}{\gamma}}.
$$

Using (A.2), we obtain:

$$
\mathbb{E}_t [U (A_T^{su})] = \frac{1}{1 - \gamma} (A_T^{su})^{1 - \gamma} G_t^{1 - \gamma} B(t, T)^{(\gamma - 1)(1 - \frac{1}{\gamma})} G_t B(t, T)^{1 - \frac{1}{\gamma}}
= \frac{1}{1 - \gamma} \left( \frac{A_T^{su}}{B(t, T)} \right)^{1 - \gamma} G_t^\gamma,
$$

which concludes the proof.

**A.2 Proof of Proposition 2**

Consider a strategy with terminal wealth $A_T$ that satisfies $A_T \geq F_T$ almost surely. The objective is to show that:

$$
\mathbb{E} [U (A_T)] \leq \mathbb{E} [U (X^*)]. \quad (A.5)
$$

The argument that we present now is borrowed from El Karoui et al. (2005), which we adapt to the case of a stochastic floor. Because $U$ is concave, we have that:

$$
U (A_T) - U (X^*) \leq \hat{U} (X^*) (A_T - X^*). \quad (A.6)
$$

Since $X^* = \max(\xi A_T^{su}, F_T)$ and marginal utility is decreasing, we have:

$$
\hat{U} (X^*) = \hat{U} (\xi A_T^{su}) - \left[ \hat{U} (\xi A_T^{su}) - \hat{U} (F_T) \right]^+
= \xi^{-\frac{1}{\gamma}} \hat{U} (A_T^{su}) - \left[ \hat{U} (\xi A_T^{su}) - \hat{U} (F_T) \right]^+.
$$

Moreover, $A^{su}_T$ is the optimal unconstrained strategy, so we have (see (A.1)):

$$
\hat{U} (A_T^{su}) = \eta^{-\gamma} M_T^{su},
$$

with a constant $\eta$. Since we have $\mathbb{E} [M_T^{su} A_T^{su}] = \mathbb{E} [M_T^{su} A_T]$, it follows that the expectation of the right side in (A.6) is:

$$
\mathbb{E} \left[ \hat{U} (X^*) (A_T - X^*) \right] = -\mathbb{E} \left[ \left[ \hat{U} (\xi A_T^{su}) - \hat{U} (F_T) \right]^+ (A_T - X^*) \right].
$$

The positive part within the expectation operator is zero, unless $\xi A_T^{su} \leq F_T$, in which case $A_T^{su}$ is equal to $F_T$. Hence:

$$
\mathbb{E} \left[ \hat{U} (X^*) (A_T - X^*) \right] = -\mathbb{E} \left[ \left[ \hat{U} (\xi A_T^{su}) - \hat{U} (F_T) \right]^+ (A_T - F_T) \right].
$$
Appendix

The expectation in the right side is nonnegative since $A_t \geq F_t$ almost surely, so the left side is nonpositive. Together with (A.6), this property implies (A.5).

A.3 Proof of Proposition 4
We start from the expression of optimal terminal wealth, (3.10). It implies that the optimal constrained wealth process is given by:

$$A_t^{*c} = \xi A_t^{*u} + F_t p (t, \xi R_t^{*u}, V_t).$$

Applying Ito’s lemma in both sides and matching the diffusion terms, we get:

$$A_t^{*c} \sigma w_t^{*c} = \xi A_t^{*u} \sigma w_t^{*u} + F_t \sigma w_t^F + F_t \left[ p_2 \xi R_t^{*u} \sigma_t^{R_t^{*u}} + \sigma V p_3 \right].$$

Note that the volatility vector $\sigma_t^{R_t^{*u}}$ is equal to $\sigma [w_t^{*u} - w_t^F]$. Hence:

$$w_t^{*c} = \frac{\xi A_t^{*u} + F_t p_2 \xi R_t^{*u}}{A_t^{*c}} \sigma w_t^{*u} + \frac{F_t}{A_t^{*c}} \sigma w_t^F + \frac{F_t}{A_t^{*c}} (\sigma' \sigma)^{-1} \sigma' \sigma V p_3$$

$$= \frac{A_t^{*c} - F_t p + F_t p_2 \xi R_t^{*u}}{A_t^{*c}} w_t^{*u} + \frac{F_t}{A_t^{*c}} \sigma w_t^F + \frac{F_t}{A_t^{*c}} (\sigma' \sigma)^{-1} \sigma' \sigma V p_3,$$

which leads to (3.13).

A.4 Proof of Proposition 5
In this proof, we write $m(t, \xi R_t^{*u}, V_t)$ for $m_t$, in order to explicitly indicate the dependence of $m_t$ upon the ratio $R_t^{*u}$ and the state variables $V_t$. $p(t, \xi R_t^{*u}, V_t)$ is the price of a put written on the underlying $R_t^{*u}$, and the volatility of $R_t^{*u}$ is independent from $R_t^{*u}$ (see (3.8)). Hence $p$ is decreasing and convex in its second argument. In particular, we have:

$$p_2 (t, \xi R_t^{*u}, V_t) \leq 0,$$

$$p_2 (t, \xi R_t^{*u}, V_t) - p_1 (t, 0, V_t) \leq p_2 (t, \xi R_t^{*u}, V_t) \xi R_t^{*u}.$$

Since the price of the put for a zero spot price is equal to 1, the second inequality implies:

$$p_2 (t, \xi R_t^{*u}, V_t) - p_2 (t, 0, V_t) \xi R_t^{*u} \leq 1.$$  

Hence $m(t, \xi R_t^{*u}, V_t)$ is comprised between 0 and 1.

The derivative of $m$ with respect to its second argument is:

$$m_2 (t, \xi R_t^{*u}, V_t) = -\xi R_t^{*u} p_2 (t, \xi R_t^{*u}, V_t),$$

which is a negative quantity. Hence $m$ is decreasing in $R_t^{*u}$. Since $m$ is nonnegative, it has a nonnegative limit $\ell$ as $R_t^{*u}$ grows to infinity. Because the put price shrinks to zero, we have:

$$\ell = - \lim_{R_t^{*u} \to \infty} [\xi R_t^{*u} p_2 (t, \xi R_t^{*u}, V_t)].$$

Let $F (t, \xi R_t^{*u}, V_t) = \xi R_t^{*u} p (t, \xi R_t^{*u}, V_t).$
Appendix

Then: \( F_2 (t, \xi R_t^{* u}, V_t) = p(t, \xi R_t^{* u}, V_t) + \xi R_t^{* u} p_2 (t, \xi R_t^{* u}, V_t) \),

hence:

\[
\lim_{R_t^{* u} \to \infty} F_2 (t, \xi R_t^{* u}, V_t) = -\ell.
\]

Assuming that \( \ell \) is positive, we would therefore obtain:

\[
\lim_{R_t^{* u} \to \infty} F (t, \xi R_t^{* u}, V_t) = -\infty.
\]

Because \( F \) is nonnegative, this cannot happen, so \( \ell \) is zero.

The fact that \( m(t, 0, V_t) \) equals one follows from the definition of \( m \) and the fact that \( p(t, 0, V_t) \) is one.

We have established properties of \( m \) considered as a function of \( R_t^{* u} \). In order to complete the proof, we have to establish the properties of \( m \) considered as a function of \( A_t^{* c} \). We note that \( A_t^{* c} \) is strictly increasing in \( R_t^{* u} \), and that the limits of \( A_t^{* c} \) when \( R_t^{* u} \) goes to zero or to infinity are \( F_t \) and infinity respectively. To see this, we can use (3.9) to write:

\[
A_t^{* c} = F_t + F_t \mathbb{E}^{Q_F} [ (\xi R_t^{* u} - 1)^+] .
\]

The second term in the right side is the price of a call written on \( R_t^{* u} \). Because the call price is strictly increasing in the spot price, \( A_t^{* c} \) is strictly increasing in \( R_t^{* u} \). Moreover, when \( R_t^{* u} \) shrinks to zero, the call price shrinks to zero as well, so \( A_t^{* c} \) converges to \( F_t \). When \( R_t^{* u} \) grows to infinity, the call price grows to infinity too, and so does \( A_t^{* c} \).

A.5 Extension to ALM

In this section, we write the ODE satisfied by the function \( C_2^f \) that appears in the optimal unconstrained ALM strategy. We do not provide the derivation here, because it is the same as that of the second ODE written in Proposition 1. In fact, the ODE verified by \( C_2^f \) is formally identical to the latter ODE, with \( \sigma_B \) being replaced by \( \sigma_L \):

\[
\dot{C}_2^f(s) = \frac{\Lambda_2 \Lambda_1}{\gamma} - \frac{1 - \gamma}{\gamma} \Lambda_2 \sigma_L(s) + \left[ \frac{1 - \gamma}{\gamma} \sigma_\lambda \Lambda_2 - \kappa \right] C_2^f(s) + \left[ \kappa \chi + \frac{1 - \gamma}{\gamma} \sigma_\lambda (\Lambda_1 - \sigma_L(s)) \right] C_3(s) + \frac{1 - \gamma}{\gamma} [\sigma_\lambda^2 - (1 - \gamma) \sigma_\lambda \sigma_B N \sigma_\lambda] C_2^f(s) C_3(s).
\]

A.6 Proof of Proposition 7

As a first observation, we note that the sequence of short-term constraints “\( F_t \leq A_t \leq C_t \) for all \( t \)” is equivalent to the constraint on terminal values “\( F_T \leq A_T \leq C_T \)”, because both the floor and the cap are replicable. This equivalence follows from the same argument as in (3.1). Thus, the static program that we solve is:

\[
\max_{A_T} \mathbb{E} [U(A_T)], \text{ subject to } \mathbb{E} [M_t^{* u} A_T] = A_0 \text{ and } F_T \leq A_T \leq C_T.
\]
Our guess for the optimal payoff is the random variable $A_T^{2c}$ defined in (4.2). We must show first that $A_T^{2c}$ is a replicable payoff, and second that it maximises expected utility.

For the first part, the conditions stated in the proposition imply that the two payoffs $[F_T - \xi' A_T^{1u}]^+$ and $[\xi' A_T^{2u} - C_T]^+$ are attainable.

The prices of these options are:
\[
\mathbb{E}_t \left[ \frac{M_T^{1u}}{M_t^{1u}} [F_T - \xi' A_T^{2u}]^+ \right] = F_t \mathbb{E}_q \left[ \left[ 1 - \xi' R_T^{1u} \right]^+ \right] = F_t p \left( t, \xi' R_t^{1u}, V_t \right),
\]
\[
\mathbb{E}_t \left[ \frac{M_T^{2u}}{M_t^{2u}} [\xi' A_T^{2u} - C_T]^+ \right] = C_t \mathbb{E}_q \left[ \left[ \xi' R_T^{1u} - 1 \right]^+ \right] = C_t c \left( t, \xi' R_t^{1u}, V_t \right).
\]

We now show that $A_T^{2c}$ is utility-maximising. Let us consider another payoff $A_T$ such that $F_T \leq A_T \leq C_T$ almost surely. The objective is to show that:
\[
\mathbb{E} \left[ U \left( A_T \right) \right] \leq \mathbb{E} \left[ U \left( A_T^{2c} \right) \right]. \tag{A.7}
\]

By concavity of $U$, we have that:
\[
U \left( A_T \right) - U \left( A_T^{2c} \right) \leq \hat{U} \left( A_T^{2c} \right) \left( A_T - A_T^{2c} \right). \tag{A.8}
\]

Moreover, we have:
\[
A_T^{2c} = \min \left[ \max \left[ \xi' A_T^{2u}, F_T \right], C_T \right],
\]

hence:
\[
\hat{U} \left( A_T^{2c} \right) = \max \left[ \min \left[ \hat{U} \left( \xi' A_T^{2u} \right), \hat{U} \left( F_T \right) \right], \hat{U} \left( C_T \right) \right]
= \hat{U} \left( \xi' A_T^{2u} \right) + \left[ \hat{U} \left( C_T \right) - \hat{U} \left( \xi' A_T^{2u} \right) \right]^+ - \left[ \hat{U} \left( \xi' A_T^{2u} \right) - \hat{U} \left( F_T \right) \right]^+.
\]

We have $\hat{U} \left( \xi' A_T^{2u} \right) = \eta' M_T^{2u}$ for a constant $\eta'$, hence:
\[
\mathbb{E} \left[ \hat{U} \left( \xi' A_T^{2u} \right) \left( A_T - A_T^{2c} \right) \right] = \eta' \mathbb{E} \left[ M_T^{2u} \left( A_T - A_T^{2c} \right) \right] = 0.
\]

Hence:
\[
\mathbb{E} \left[ \hat{U} \left( A_T^{2c} \right) \left( A_T - A_T^{2c} \right) \right]
= \mathbb{E} \left[ \left[ \hat{U} \left( C_T \right) - \hat{U} \left( \xi' A_T^{2u} \right) \right]^+ - \left[ \hat{U} \left( \xi' A_T^{2u} \right) - \hat{U} \left( F_T \right) \right]^+ \right] \left( A_T - A_T^{2c} \right) \left( A_T - A_T^{2c} \right). \tag{A.9}
\]

It can be checked that in all states of the world, we have $X_T \leq 0$. Hence:
\[
\mathbb{E} \left[ \hat{U} \left( A_T^{2c} \right) \left( A_T - A_T^{2c} \right) \right] \leq 0.
\]

Taking expectations on both sides of (A.8), we arrive at (A.7).
The value of the payoff (4.2) at date $t$ is:

$$A_t^{2c} = \xi'A_t^{su} + F_t \rho(t, \xi'R_t^{su}, V_t) - C_t c(t, \xi'R_t^{su}, V_t).$$

Applying Ito's lemma and identifying diffusion terms in both sides of this equality, we obtain:

$$A_t^{2c}w_t^{su} = \xi'A_t^{su}w_t^{su} + F_t \rho w_t^F + F_t \left[ p_2 \xi'R_t^{su} (w_t^{su} - w_t^F) + (\sigma'\sigma)^{-1} \sigma'\sigma^V p_3 \right] - C_t c w_t^{2c} - C_t [2\xi'R_t^{su} (w_t^{su} - w_t^C) + (\sigma'\sigma)^{-1} \sigma'\sigma^V c_3].$$

Letting $m_t' = p - \xi'R_t^{su}p_2$ and $m_t'' = c - \xi'R_t^{su}c_2$ and rearranging terms, we obtain $(4.4)$.

We now examine the cost of insurance in payoff $(4.2)$. This cost is equal to the price of the synthetic option strategy that pays $[F_T - \xi'A_T^{su}]^+$ at date $T$. By definition of $\xi'$, this price equals $(1 - \xi')A_0$. Using put-call parity, we can rewrite $(4.2)$ as:

$$A_T^{2c} = F_T + [\xi'A_T^{su} - F_T]^+ - [\xi'A_T^{su} - C_T]^+$$

Multiplying both sides by $M_T^{su}$ and taking expectations, we obtain:

$$A_0 - F_0 = F_0 \mathbb{E}^Q \left[ [\xi'R_T^{su} - 1]^+ \right] - c(0, \xi'R_0^{su}, V_0).$$

Moreover, by definition of $\xi$, we have:

$$A_0 - F_0 = F_0 \mathbb{E}^Q \left[ [\xi'R_T^{su} - 1]^+ \right]$$

The call price $c(0, \xi'R_0^{su}, V_0)$ is positive, because there is a positive probability (under $\mathbb{P}$ and under $Q^F$) for $\xi'R_T^{su}$ to end up above 1. Hence:

$$\mathbb{E}^Q \left[ [\xi'R_T^{su} - 1]^+ \right] > \mathbb{E}^Q \left[ [\xi'R_T^{su} - 1]^+ \right].$$

Since the call price is strictly increasing in the spot price of the underlying, we obtain $\xi'R_0^{su} > \xi R_0^{su}$, hence $\xi' > \xi$.

A.7 Wealth Generated by FM Strategies

Using Ito's lemma, it is straightforward to show that the wealth $A_t^{fm}$ generated by a self-financing portfolio following the fixed-mix strategy:

$$w_t = aw_t^o + bw_t^b,$$

where $w_t^o$ generates wealth $A_t$ and $w_t^b$ wealth $B_t$, is given by

$$A_t^{fm} = A_0 A_t^{B_t} B_t^{A_t} \exp \left[ (1 - a - b) \int_0^t r_s ds + \frac{1}{2} \alpha(1 - a) \int_0^t \| \sigma_s w_s^o \|^2 ds + \frac{1}{2} b(1 - b) \int_0^t \| \sigma_s w_s^b \|^2 ds - ab \int_0^t (w_s^o)' \sigma_s \sigma_s w_s^b ds \right].$$

Then, if we consider a self-financing portfolio evolving with the FM strategy given in (6.2), we can derive its value $A_t^{fm}$ by using (A.10) with $a = x$ and $b = 1 - x$. After simplification, we obtain

$$A_t^{fm} = A_0 S_t^F \frac{F_t^{1-x}}{S_t^F F_0^{1-x}} \exp \left[ \frac{1}{2} x(1 - x) \int_0^t \| \sigma_s (w_s^S - w_s^F) \|^2 ds \right].$$
### Appendix

#### A.8 Wealth Generated by CPPI Strategies

We consider a self-financing portfolio evolving with the portfolio strategy (6.3), and note that the dynamics of the risk budget reads:

\[
d \left[ A_t^{\text{ppi}} - F_t \right] = dA_t^{\text{ppi}} - dF_t \\
= \left[ A_t^{\text{ppi}} - F_t \right] r_t \, dt + \left[ A_t^{\text{ppi}} w_t^{\text{ppi}} - F_t w_t^F \right]' \sigma_t' \left[ \lambda \, dt + dz_t \right] \\
= \left[ A_t^{\text{ppi}} - F_t \right] \left[ r_t \, dt + \left[ m \, w_t^S + (1-m) \, w_t^F \right]' \sigma_t' \left[ \lambda \, dt + dz_t \right] \right],
\]

which proves that the risk budget can be regarded as the value of a fixed-mix strategy invested in the stock and the floor-replicating portfolio. Therefore, we can use (A.10) with \( a = m \) and \( b = 1 - m \) and conclude that:

\[
A_t^{\text{ppi}} - F_t = (A_0 - F_0) \frac{S_t^m}{S_0^m} F_t^{1-m} \exp \left[ \frac{1}{2} m(1-m) \int_0^t \| \sigma_s (w_s^S - w_s^F) \|^2 \, ds \right].
\]

From this wealth dynamic, one can easily verify that the wealth generated by the CPPI strategy is always above the floor \( F_t \) at any time \( t \leq T \).

#### B. Tables and Figures

*Table 1: Maximum Likelihood Estimates for Numerical Illustration 1.*

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<th>Parameter</th>
<th>Estimate</th>
<th>Standard deviation</th>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlations</th>
<th>Estimate</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^{xS} )</td>
<td>-0.0118</td>
<td>0.0685</td>
</tr>
<tr>
<td>( \rho^{x\lambda} )</td>
<td>0.0631</td>
<td>0.0961</td>
</tr>
<tr>
<td>( \rho^{\lambda S} )</td>
<td>-0.0494</td>
<td>0.0890</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volatilities of measurement errors on bond yields</th>
<th>Estimate</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma^{xY} )</td>
<td>0.0062</td>
<td>&lt; 0.0001</td>
</tr>
<tr>
<td>( \sigma^{xY} )</td>
<td>0.0080</td>
<td>&lt; 0.0001</td>
</tr>
<tr>
<td>( \sigma^{xY} )</td>
<td>0.0101</td>
<td>&lt; 0.0001</td>
</tr>
<tr>
<td>( \sigma^{xY} )</td>
<td>0.0117</td>
<td>&lt; 0.0001</td>
</tr>
</tbody>
</table>

Estimates for all parameter values have been obtained by maximising the log-likelihood of quarterly observations on US market. Standard deviations have been computed by taking the square roots of diagonal elements of the inverse of Fisher’s information matrix.
Appendix

Figure 1: Cost of Optimally Managing the Insurance as a Function of the Risk Aversion.

These figures display the cost $1 - \xi$ of the insurance as a function of the risk aversion $\gamma$. The three colors on both graphs represent three different initial values for the floor expressed as a percentage of the initial wealth $A_0$. The panel on the top shows the results for an investment horizon of $T = 10Y$, while the panel at the bottom shows the results for an investment horizon of $T = 20Y$. 
Appendix

Figure 2: Cost of Optimally Managing the Insurance as a Function of the Initial Value of the Floor.

These figures display the cost $1 - \xi$ of the insurance as a function of the ratio of the floor $F_0$ to the wealth $A_0$ at time 0. The three colors on both graphs represent three different risk aversions. These levels are calibrated in such a way that the average allocation to stocks in the unconstrained strategy, computed with the longest horizon (i.e. $T = 20Y$) and across all 30,000 simulated paths, is equal to 75% for the aggressive, 50% for the moderate, and 25% for the defensive. The panel on the top shows the results for an investment horizon of $T = 10Y$, while the panel at the bottom shows the results for an investment horizon of $T = 20Y$. 
Appendix

Figure 3: Monetary Utility Losses of Optimal Insurance as a Function of the Risk Aversion.

These figures display the Monetary Utility Losses (MUL) of introducing an optimal management of the insurance into the strategy as a function of the risk aversion $\gamma$. The three colors on both graphs represent three different initial values for the floor expressed as a percentage of the initial wealth $A_0$. The panel on the top shows the results for an investment horizon of $T = 10Y$, while the panel at the bottom shows the results for an investment horizon of $T = 20Y$. The MULs are computed with respect to the optimal unconstrained strategy implemented in continuous time.
Appendix

Figure 4: Monetary Utility Losses of Optimal Insurance as a Function of the Initial Value of the Floor.

These figures display the Monetary Utility Losses (MUL) of introducing an optimal management of the insurance into the strategy as a function of the ratio of the floor $F_0$ to the wealth $A_0$ at time 0. The three colors on both graphs represent three different risk aversions. These levels are calibrated in such a way that the average allocation to stocks in the unconstrained strategy, computed with the longest horizon (i.e. $T = 20Y$) and across all 30,000 simulated paths, is equal to 75% for the aggressive, 50% for the moderate, and 25% for the defensive. The panel on the top shows the results for an investment horizon of $T = 10Y$, while the panel at the bottom shows the results for an investment horizon of $T = 20Y$. The MULs are computed with respect to the optimal unconstrained strategy implemented in continuous time.
Appendix

Figure 5: Impact of Equity Sharpe Ratio Variability on Monetary Utility Losses of Strategies Implemented with Optimal Insurance.

These figures display the Monetary Utility Losses (MUL) of an optimal management of the insurance into the strategy as a function of the ratio of the floor $F_0$ to the wealth $A_0$ at time 0. The three colors on both graphs represent three different risk aversions. These levels are calibrated in such a way that the average allocation to stocks in the unconstrained strategy, computed with the longest horizon (i.e. $T = 20Y$) and across all 30,000 simulated paths, is equal to 75% for the aggressive, 50% for the moderate, and 25% for the defensive. The panel on the top shows the results for a low variability of the equity Sharpe ratio $\sigma_\lambda = 10\%$, while the panel at the bottom shows the results for a very low variability of the equity Sharpe ratio $\sigma_\lambda = 5\%$. The MULs are computed with respect to the optimal unconstrained strategy implemented in continuous time.
Appendix

Figure 6: Impact of Adding a Cap in the Strategy to Decrease the Cost of Insurance.

The top panel displays the cost of the insurance $1 - \xi$ when the wealth has upper and lower bounds, as a function of the ratio of the cap $C_0$ to the wealth $A_0$ at time 0. The bottom panel displays the Monetary Utility Losses (MUL) of an optimal management of the insurance with upper and lower bounds as a function of the ratio of the cap $C_0$ to the wealth $A_0$ at time 0. The three colors on both graphs represent three different risk aversions. These levels are calibrated in such a way that the average allocation to stocks in the unconstrained strategy, computed with the longest horizon (i.e. $T = 20Y$) and across all 30,000 simulated paths, is equal to 75% for the aggressive, 50% for the moderate, and 25% for the defensive. The MULs are computed with respect to the optimal unconstrained strategy implemented in continuous time, and with an initial value of the floor $F_0$ equal to 90% of $A_0$. 
Appendix

Figure 7: Monetary Utility Losses of Inappropriate CPPI-like Insurance as a Function of the Initial Value of the Floor.

These figures display the Monetary Utility Losses (MUL) of introducing an inappropriate CPPI-like insurance into the strategy as a function of the ratio of the floor $F_0$ to the wealth $A_0$ at time $t$. For each CPPI strategy (classical CPPI and extended CPPI), we consider five possible values $m = 1, ..., 5$. The three panels represent three different risk aversions. The panel on top shows the results for an aggressive investor, while the panel in the middle shows the results for a moderate investor, and the panel at the bottom represents a defensive investor. The risk aversion of these investors is calibrated in such a way that the average allocation to stocks in the unconstrained strategy, computed with the longest horizon (i.e. $T = 20Y$) and across all 30,000 simulated paths, is equal to 75% for the aggressive, 50% for the moderate, and 25% for the defensive. The MULs are computed with respect to the optimal unconstrained strategy implemented in continuous time.
Figure 8: Monetary Utility Losses of Inappropriate FM-like Insurance as a Function of the Initial Value of the Floor.

These figures display the Monetary Utility Losses (MUL) of introducing an inappropriate FM-like insurance into the strategy as a function of the ratio of the floor $F_0$ to the wealth $A_0$ at time 0. The three panels represent three different risk aversions. The panel on the top shows the results for an aggressive investor, while the panel in the middle shows the results for a moderate investor, and the panel at the bottom shows those for a defensive investor. The risk aversion of these investors are calibrated in such a way that the average allocation to stocks in the unconstrained strategy, computed with the longest horizon (i.e. $T = 20Y$) and across all 30,000 simulated paths, is equal to 75% for the aggressive, 50% for the moderate, and 25% for the defensive. The MULs are computed with respect to the optimal unconstrained strategy implemented in continuous time.
Table 2: Base Case Parameter Values for Numerical Illustration 2.

<table>
<thead>
<tr>
<th>Nominal short-term rate.</th>
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</thead>
<tbody>
<tr>
<td>$dr_t = (b - \lambda_t)dt + \sigma_r , dz_r^f$.</td>
</tr>
<tr>
<td>$b$</td>
</tr>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\sigma_r$</td>
</tr>
<tr>
<td>$\lambda_r$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance of stock index.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_t = (\sigma_t^S)^2$, $dV_t = (\lambda_t^V - V_t) , dt + \sigma^V \sqrt{V_t} , dz_t^V$.</td>
</tr>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\sigma_V$</td>
</tr>
<tr>
<td>$\sqrt{\lambda}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sharpe ratio of stock index.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d\lambda_t^S = \langle \lambda - \lambda_t^S \rangle , dt + \sigma_{\lambda} , dz_t^\lambda$.</td>
</tr>
<tr>
<td>$\kappa$</td>
</tr>
<tr>
<td>$\lambda$</td>
</tr>
<tr>
<td>$\sigma_{\lambda}$</td>
</tr>
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<table>
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<tr>
<th>Correlations.</th>
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<tr>
<td>$\rho_{SA}$</td>
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<tr>
<td>$\rho_{SR}$</td>
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<tr>
<td>$\rho_{SV}$</td>
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<tr>
<td>$\rho_{SV}$</td>
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<table>
<thead>
<tr>
<th>Initial Values.</th>
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<td>$S_0$</td>
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<td>$\lambda_0$</td>
</tr>
<tr>
<td>$\tau_0$</td>
</tr>
<tr>
<td>$V_0$</td>
</tr>
</tbody>
</table>

This table displays the base case values used for illustrating the benefits of max drawdown constraints (see Section 7). These parameter values come from Amenc et al. (2012).
Appendix

Figure 9: Distributions of Terminal Wealth Generated by Long-Term Investment Strategies.

Panel (a) displays distributions of terminal wealth generated by long-term unconstrained strategies. The three different risk aversion roughly correspond to an average stock allocation of 30% (aggressive strategy with $\gamma = 12$), 20% (moderate strategy with $\gamma = 23$), and 10% (defensive strategy with $\gamma = 55$) over a 20-year investment period. Panel (b) displays distributions of terminal wealth generated by long-term constrained strategies when various maximum drawdown constraints are imposed.
Table 3: Percentiles, Risk and Dispersion Measures of Terminal Wealth for Long-Term Unconstrained Strategies.

<table>
<thead>
<tr>
<th></th>
<th>Aggressive</th>
<th>Moderate</th>
<th>Defensive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min Wealth</td>
<td>251.71</td>
<td>280.10</td>
<td>293.75</td>
</tr>
<tr>
<td>Q5</td>
<td>362.39</td>
<td>345.16</td>
<td>324.78</td>
</tr>
<tr>
<td>Low Target Wealth (Q25)</td>
<td>419.64</td>
<td>378.00</td>
<td>338.89</td>
</tr>
<tr>
<td>Medium Target Wealth (Q50)</td>
<td>459.49</td>
<td>399.71</td>
<td>348.11</td>
</tr>
<tr>
<td>High Target Wealth (Q75)</td>
<td>500.33</td>
<td>421.75</td>
<td>357.40</td>
</tr>
<tr>
<td>Q95</td>
<td>567.38</td>
<td>456.22</td>
<td>371.75</td>
</tr>
<tr>
<td>Max Wealth</td>
<td>739.16</td>
<td>540.35</td>
<td>403.29</td>
</tr>
<tr>
<td>Average Wealth</td>
<td>461.37</td>
<td>400.15</td>
<td>348.18</td>
</tr>
<tr>
<td>High minus Low</td>
<td>80.49</td>
<td>43.75</td>
<td>18.51</td>
</tr>
<tr>
<td>(High minus Low)/(2 × Medium)</td>
<td>8.78%</td>
<td>5.47%</td>
<td>2.66%</td>
</tr>
<tr>
<td>Max 3Y-loss</td>
<td>15.89%</td>
<td>9.26%</td>
<td>7.75%</td>
</tr>
<tr>
<td>Max DD</td>
<td>24.40%</td>
<td>17.78%</td>
<td>15.16%</td>
</tr>
</tbody>
</table>

This table displays various percentiles of the distribution of terminal wealth generated by the long-term unconstrained strategies, together with measures of dispersion and risk. The maximum 3Y-loss and the maximum drawdown have been computed by discarding 0.5% of the worst 3Y-loss, and 0.5% of the worst maximum drawdown respectively, so as to robustify the estimation. Three levels of risk aversion are considered. Short-sale constraints are imposed, and parameters are fixed at their base case values (see Table 2).

Table 4: Opportunity Cost of Selecting a More Conservative Long-Term Unconstrained Strategy in % of Initial Wealth.

<table>
<thead>
<tr>
<th>Higher γ</th>
<th>Lower γ Moderate</th>
<th>Aggressive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Defensive</td>
<td>14.93</td>
<td>32.51</td>
</tr>
<tr>
<td>Moderate</td>
<td>-</td>
<td>15.30</td>
</tr>
</tbody>
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This table displays the increase in initial contribution that is required from the investor with the higher risk aversion in order to reach the same average wealth level as the investor with the lower risk aversion. Both investors implement a long-term unconstrained strategy. Parameters are fixed at their base case values (see Table 2).

Table 5: Percentiles and Short-Term Measures of Risk for Long-Term Strategies with Drawdown Constraints.

<table>
<thead>
<tr>
<th></th>
<th>Aggressive with 20% Max DD</th>
<th>Aggressive with 15% Max DD</th>
<th>Aggressive with 10% Max DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q5</td>
<td>353.57</td>
<td>338.73</td>
<td>279.11</td>
</tr>
<tr>
<td>Low Target Wealth (Q25)</td>
<td>413.85</td>
<td>394.25</td>
<td>324.33</td>
</tr>
<tr>
<td>Medium Target Wealth (Q50)</td>
<td>454.65</td>
<td>436.54</td>
<td>358.61</td>
</tr>
<tr>
<td>High Target Wealth (Q75)</td>
<td>496.55</td>
<td>478.96</td>
<td>396.70</td>
</tr>
<tr>
<td>Q95</td>
<td>564.05</td>
<td>545.98</td>
<td>458.03</td>
</tr>
<tr>
<td>Average Wealth</td>
<td>456.05</td>
<td>437.82</td>
<td>362.46</td>
</tr>
<tr>
<td>Max 3Y-loss (%)</td>
<td>14.03</td>
<td>10.82</td>
<td>6.17</td>
</tr>
<tr>
<td>Max DD (%)</td>
<td>18.74</td>
<td>14.52</td>
<td>9.66</td>
</tr>
</tbody>
</table>

This table displays various percentiles of the distribution of terminal wealth together with short-term measures of risk for three long-term constrained strategies implemented with various drawdowns. The maximum 3Y-loss and the maximum drawdown have been computed by discarding 0.5% of the worst 3Y-loss, and 0.5% of the worst maximum drawdown respectively. Short-sale constraints are imposed, and parameters are fixed at their base case values (see Table 2).
## Appendix

Table 6: Opportunity Cost of Imposing a Stricter Drawdown Constraint in % of Initial Wealth.

<table>
<thead>
<tr>
<th></th>
<th>Aggressive</th>
<th>Moderate</th>
<th>Defensive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15% Max DD</td>
<td>20% Max DD</td>
<td>No Max DD</td>
</tr>
<tr>
<td>Looser constraint</td>
<td>10% Max DD</td>
<td>17.43</td>
<td>13.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>19.58</td>
<td>15.14</td>
</tr>
<tr>
<td></td>
<td>15% Max DD</td>
<td>1.83 %</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.05</td>
<td>1.11</td>
</tr>
<tr>
<td></td>
<td>20% Max DD</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.22 %</td>
<td>-</td>
</tr>
</tbody>
</table>

This table displays the increase in the initial contribution that is required from the investor who wants to impose a strict maximum drawdown constraint in order to reach the same average wealth level as an investor who is less insured against drawdown risk. Parameters are fixed at their base case values (see Table 2).
Appendix

Figure 10: Evolution of the Allocation over Time

Panels (a) and (b) present the allocation result of an empirical test with the MSCI EMU as a proxy for the equity index. The investment horizon is equal to 15 years, and we consider an aggressive investor investing in average 30% of his wealth in the equity index. Panel (a) displays, in the absence of drawdown constraint, the evolution of the equity index, the LHP, and the cash investments over time. Panel (b) displays, in the presence of a 15% max drawdown constraint, the evolution of the equity index, the LHP, and the cash investments over time.
References
References

References


References

BNP Paribas Investment Partners is the dedicated asset management business line of the BNP Paribas Group. BNP Paribas Investment Partners offers a full range of investment management products and services to institutional and corporate, as well as retail clients across the globe.

With total assets under management of EUR 503 billion (USD 664 billion) as of 31 December 2012, BNP Paribas Investment Partners is the sixth-largest asset manager in Europe and amongst the twenty largest asset managers worldwide. BNP Paribas Investment Partners has asset management capabilities in 40 countries, and distribution in over 70 countries. It has offices in all of the world’s major financial centres, including Hong Kong, London, New York, Paris and Tokyo.

Central to the way we work is the concept of partnership and close relationships, both with our clients and across our network of some 60 investment centres, where about 800 investment professionals work on all major asset classes, each a specialist in a given strategy, asset class or type of product.

BNP Paribas Investment Partners combines the financial strength, powerful distribution and compliance focus of its parent company with the reactivity, specialisation and entrepreneurial spirit of smaller boutiques.

BNP Paribas Investment Partners provides a broad range of expertise, both for global and local solutions from its various affiliates.

To name the main capabilities and entities:
- Fundamental management of all major assets classes (BNP Paribas Asset Management)
- Global and emerging fixed income (FFTW)
- Quantitative global equity (Alfred Berg)
- Multi-management Advisory (FundQuest Advisor)
- Environmental markets (Impax Asset Management)
- Private portfolio management (CamGestion, BNP Paribas Discretionary Portfolio Management)
- Employee retirement and saving schemes (BNP Paribas Epargne et Retraite Entreprises)
- Private equity (BNP Paribas Private Equity)
- Australia (Arnhem Investment Management)
- Brazil (BNP Paribas Asset Management Brasil)
- Chile (BancoEstado Administradora General de Fondos)
- China (HFT Investment Management Co. Ltd.)
- India (BNP Paribas Mutual Funds)
- Indonesia (PT. BNP Paribas Investments Partners Indonesia)
- Morocco (BMCI Gestion)
- Russia (TKB BNP Paribas Investments Partners)
- South Korea (Shinhan BNP Paribas AMC)
- Turkey (TEB)

A signatory of the Principles for Responsible Investment (“PRI”), BNP Paribas has been constantly reaffirming its commitment as a responsible investor and, in 2012, included the application of ESG criteria to all of its open-ended funds. At the same time, BNP Paribas has formalised its own Corporate Social Responsibility principles in an explicit charter.
About EDHEC-Risk Institute
About EDHEC-Risk Institute

The Choice of Asset Allocation and Risk Management
EDHEC-Risk structures all of its research work around asset allocation and risk management. This strategic choice is applied to all of the Institute’s research programmes, whether they involve proposing new methods of strategic allocation, which integrate the alternative class; taking extreme risks into account in portfolio construction; studying the usefulness of derivatives in implementing asset-liability management approaches; or orienting the concept of dynamic “core-satellite” investment management in the framework of absolute return or target-date funds.

Academic Excellence and Industry Relevance
In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by the Institute’s advisory board. This board is made up of internationally recognised researchers, the Institute’s business partners, and representatives of major international institutional investors. Management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Six research programmes have been conducted by the centre to date:
• Asset allocation and alternative diversification
• Style and performance analysis
• Indices and benchmarking
• Operational risks and performance
• Asset allocation and derivative instruments
• ALM and asset management

These programmes receive the support of a large number of financial companies. The results of the research programmes are disseminated through the EDHEC-Risk locations in Singapore, which was established at the invitation of the Monetary Authority of Singapore (MAS); the City of London in the United Kingdom; Nice and Paris in France; and New York in the United States.

EDHEC-Risk has developed a close partnership with a small number of sponsors within the framework of research chairs or major research projects:
• Core-Satellite and ETF Investment, in partnership with Amundi ETF
• Regulation and Institutional Investment, in partnership with AXA Investment Managers
• Asset-Liability Management and Institutional Investment Management, in partnership with BNP Paribas Investment Partners
• Risk and Regulation in the European Fund Management Industry, in partnership with CACEIS
• Exploring the Commodity Futures Risk Premium: Implications for Asset Allocation and Regulation, in partnership with CME Group
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- Asset-Liability Management Techniques for Sovereign Wealth Fund Management, *in partnership with Deutsche Bank*
- The Benefits of Volatility Derivatives in Equity Portfolio Management, *in partnership with Eurex*
- Structured Products and Derivative Instruments, *sponsored by the French Banking Federation (FBF)*
- Optimising Bond Portfolios, *in partnership with the French Central Bank (BDF Gestion)*
- Asset Allocation Solutions, *in partnership with Lyxor Asset Management*
- Infrastructure Equity Investment Management and Benchmarking, *in partnership with Meridiam and Campbell Lutyens*
- Investment and Governance Characteristics of Infrastructure Debt Investments, *in partnership with Natixis*
- Advanced Modelling for Alternative Investments, *in partnership with Newedge Prime Brokerage*
- Advanced Investment Solutions for Liability Hedging for Inflation Risk, *in partnership with Ontario Teachers’ Pension Plan*
- The Case for Inflation-Linked Corporate Bonds: Issuers’ and Investors’ Perspectives, *in partnership with Rothschild & Cie*
- Solvency II, *in partnership with Russell Investments*
- Structured Equity Investment Strategies for Long-Term Asian Investors, *in partnership with Société Générale Corporate & Investment Banking*

The philosophy of the Institute is to validate its work by publication in international academic journals, as well as to make it available to the sector through its position papers, published studies, and conferences.

Each year, EDHEC-Risk organises three conferences for professionals in order to present the results of its research, one in London (EDHEC-Risk Days Europe), one in Singapore (EDHEC-Risk Days Asia), and one in New York (EDHEC-Risk Days North America) attracting more than 2,500 professional delegates.

EDHEC also provides professionals with access to its website, www.edhec-risk.com, which is entirely devoted to international asset management research. The website, which has more than 58,000 regular visitors, is aimed at professionals who wish to benefit from EDHEC’s analysis and expertise in the area of applied portfolio management research. Its monthly newsletter is distributed to more than 1.5 million readers.

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<th>EDHEC-Risk Institute: Key Figures, 2011-2012</th>
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<tr>
<td>Nbr of permanent staff</td>
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<td>Nbr of research associates</td>
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<td>Nbr of affiliate professors</td>
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<tr>
<td>Overall budget</td>
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<td>External financing</td>
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<tr>
<td>Nbr of conference delegates</td>
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<td>Nbr of participants at research seminars</td>
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<tr>
<td>Nbr of participants at EDHEC-Risk Institute Executive Education seminars</td>
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About EDHEC-Risk Institute

The EDHEC-Risk Institute PhD in Finance

The EDHEC-Risk Institute PhD in Finance is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world’s best universities, such as Princeton, Wharton, Oxford, Chicago and CalTech, and enjoying the support of the research centre with the greatest impact on the financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

Research for Business

The Institute’s activities have also given rise to executive education and research service offshoots. EDHEC-Risk’s executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes. In partnership with CFA Institute, it has developed advanced seminars based on its research which are available to CFA charterholders and have been taking place since 2008 in New York, Singapore and London.

As part of its policy of transferring know-how to the industry, EDHEC-Risk Institute has also set up ERI Scientific Beta. ERI Scientific Beta is an original initiative which aims to favour the adoption of the latest advances in smart beta design and implementation by the whole investment industry. Its academic origin provides the foundation for its strategy: offer, in the best economic conditions possible, the smart beta solutions that are most proven scientifically with full transparency in both the methods and the associated risks.

In 2012, EDHEC-Risk Institute signed two strategic partnership agreements with the Operations Research and Financial Engineering department of Princeton University to set up a joint research programme in the area of risk and investment management, and with Yale School of Management to set up joint certified executive training courses in North America and Europe in the area of investment management.

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- Padmanaban, N., M. Mukai, L. Tang, and V. Le Sourd. Assessing the quality of Asian stock market indices (February).
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